TOROIDAL VARIETIES AND THE WEAK FACTORIZATION THEOREM

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ABSTRACT. We develop the theory of stratified toroidal varieties, which gives, together with the theory of birational cobordisms [73], a proof of the weak factorization conjecture for birational maps in characteristic zero: a birational map between complete nonsingular varieties over an algebraically closed field K of characteristic zero is a composite of blow-ups and blow-downs with smooth centers.

CONTENTS

0. INTRODUCTION

The main goal of the present paper is two-fold. First we extend the theory of toroidal embeddings introduced by Kempf, Knudsen, Mumford, and Saint-Donat to the class of toroidal varieties with stratifications. Second we give a proof of the following weak factorization conjecture as an application and illustration of the theory.

Conjecture 0.0.1. The Weak Factorization conjecture

1. *Lef $f : X--\rightarrow Y$ be a birational map of smooth complete varieties over an algebraically closed field of characteristic zero, which is an isomorphism over an open set U. Then f can be factored as*

$$X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \ldots \xrightarrow{f_{n-1}} X_n = Y,$$

   where each X_i is a smooth complete variety and f_i is a blow-up or blow-down at a smooth center which is an isomorphism over U.
2. *Moreover, if $X \setminus U$ and $Y \setminus U$ are divisors with normal crossings, then each $D_i := X_i \setminus U$ is a divisor with normal crossings and f_i is a blow-up or blow-down at a smooth center which has normal crossings with components of D_i.*

This theorem extends a theorem of Zariski, which states that any birational map between two smooth complete surfaces can be factored into a succession of blow-ups at points followed by a succession of blow-downs at points. A stronger version of the above theorem, called the strong factorization conjecture remains open.

Conjecture 0.0.2. Strong factorization conjecture. *Any birational map $f : X \dashrightarrow Y$ of smooth complete varieties can be factored into a succession of blow-ups at smooth centers followed by a succession of blow-downs at smooth centers.*

One can find the statements of both conjectures in many papers. Hironaka [29] formulated the strong factorization conjecture. The weak factorization problem was stated by Miyake and Oda [46]. The toric versions of the strong and weak factorizations were also conjectured by Miyake and Oda [46] and are called the strong and weak Oda conjectures. The 3-dimensional toric version of the weak form was established by Danilov [21] (see also Ewald [24]). The weak toric conjecture in arbitrary dimensions was proved in [72] and later independently by Morelli [49], who proposed a proof of the strong factorization conjecture (see also Morelli[50]). Morelli's proof of the weak Oda conjecture was completed, revised and generalized to the toroidal case by Abramovich, Matsuki and Rashid in [5]. A gap in Morelli's proof of the strong Oda conjecture, which was not noticed in [5], was later found by K. Karu.

The local version of the strong factorization problem was posed by Abhyankar in dimension 2 and by Christensen in general, who has solved it for 3-dimensional toric varieties [13]. The local version of the weak factorization problem (in characteristic 0) was solved by Cutkosky in [16], who also showed that Oda's strong conjecture implies the local version of the strong conjecture for proper birational morphisms [17] and proved the local strong factorization conjecture in dimension 3 ([17]) via Christensen's theorem.

Shortly after the present proof was found, another proof of the weak factorization conjecture was conceived by Abramovich, Karu, Matsuki and the author [4]. Unlike the proof in [4] the present proof does not refer to the weak factorization theorem for toric varieties.

Another application of the theory of stratified toroidal varieties is given in Section 8.3, where we show the existence of a resolution of singularities of toroidal varieties in arbitrary characteristic by blowing up ideals determined by valuations (see Theorem 8.3.2).

All schemes and varieties in the present paper are considered over an algebraically closed field K. The assumption of characteristic 0 is needed for the results in Sections 11.4 and 12.4 only, where we use Hironaka's canonical resolution of singularities and canonical principalization for the second part of the theorem (see Hironaka [29], Villamayor [71] and Bierstone-Milman [8]).

1. MAIN IDEAS

1.1. **Toroidal embeddings.** The theory of toroidal embeddings was introduced and developed by Kempf, Knudsen, Mumford and Saint-Donat in [39]. Although it was originally conceived for the purpose of compactifying symmetric spaces and certain moduli spaces it has been successfully applied to various problems concerning resolution of singularities and factorization of morphisms (see [39], [2], [3]). It is an efficient tool in many situations, turning complicated or even "hopeless" algebro-geometric problems into relatively "easy" combinatorial problems. Its significance lies in a simple combinatorial description of the varieties considered carrying rich and precise information about singularities and stratifications.

By *a toroidal embedding* (originally a toroidal embedding without self-intersections) we mean a variety X with an open subset U with the following property: for any $x \in X$ there is an open neighborhood U_x and an étale morphism $\phi : U_x \to X_{\sigma_x}$ into a toric variety X_{σ_x} containing a torus T such that $U_x \cap U = \phi^{-1}(T)$. Such a morphism is called a *chart*.

Let $D_i, i \in I$, be the irreducible components of the complement divisor $D = X \setminus U$. We define a stratification S on X with strata $s \in S$ such that either $\overline{s}$ is an irreducible component of $\bigcap_{i \in J} D_i$ for some $J \subset I$ or $\overline{s} = X$. The strata are naturally ordered by the following relation: $s \leq s'$ iff the closure $\overline{s}$ contains s'. In particular $s = \overline{s} \setminus \bigcup_{s' > s} \overline{s'}$. Any chart $\phi : U_x \to X_{\sigma_x}$ preserves strata: as a matter of fact, all strata $s \subset U_x$ for $s \in S$ are preimages of T-orbits on X_{σ_s}. In fact a toroidal embedding can be defined as a stratified variety such that for any point $x \in X$ there exists a neighborhood U_x of x and an étale morphism $\phi : U_x \to X_{\sigma_x}$ preserving strata.

The cone σ_x in the above definition can be described using Cartier divisors. Let x belong to a stratum $s \in S$. We associate with $s \in S$ the following data ([39]):

$\mathbf{M}^s$: the group of Cartier divisors in $\mathrm{Star}(s,S) := \bigcup_{s' \leq s} s'$, supported in $\mathrm{Star}(s,S) \setminus U$,

$\mathbf{N}_s := \mathrm{Hom}(\mathbf{M}_s, \mathbb{Z})$,

$\mathbf{M}^s_+ \subset \mathbf{M}^s$: effective Cartier divisors,

$\sigma_s \subset \mathbf{N}^s_{\mathbb{R}}$: the dual of $\mathbf{M}^s_+$.

This gives a correspondence between strata $s \in S$ and cones. If $s' \leq s$ then $\sigma_{s'}$ is a face of σ_s. By glueing cones σ_s along their subfaces we construct a *conical complex* Σ. In contrast to in the standard theory of complexes two faces might share a few common faces, not just one. For any stratum $s \in S$ the integral vectors in the relative interior of σ_s define monomial valuations on X_{σ_s}. These valuations induce, via the morphisms ϕ monomial valuations centered at $\overline{s}$ (independent of charts).

The natural class of morphisms of toroidal embeddings are called *toroidal morphisms* and are exactly those which are locally determined by charts ϕ and toric morphisms:

A birational morphism of toroidal embeddings $f : (Y, U) \to (X, U)$ is *toroidal* if for any $x \in s \subset X$ there exists a chart $x \in U_x \to X_{\Delta^\sigma}$ and a subdivision Δ^σ of σ_s and a fiber square of morphisms of stratified varieties

$$
\begin{array}{ccc}
(U_x, U_x \cap U) & \longrightarrow & (X_{\sigma_s}, T) \\
\uparrow f & & \uparrow \\
U_x \times_{X_{\sigma_s}} X_{\Delta^\sigma} \quad \simeq \quad (f^{-1}(U_x), f^{-1}(U_x) \cap U) & \to & (X_{\Delta^\sigma}, T)
\end{array}
$$

Note that toroidal morphisms are well defined and do not depend upon the choice of the charts. This fact can be described nicely using the following Hironaka condition:

For any points x, y which are in the same stratum every isomorphism $\alpha : \widehat{X}_x \to \widehat{X}_y$ preserving stratification can be lifted to an isomorphism $\alpha' : Y \times_X \widehat{X}_x \to Y \times_X \widehat{X}_y$ preserving stratification.

Birational toroidal morphisms with fixed target are in bijective correspondence with subdivisions of the associated complex.

1.2. **Stratified toroidal varieties.**

A more general class of objects occurring naturally in applications is the class of stratified toroidal varieties defined as follows.

A *stratified toric variety* is a toric variety with an invariant equisingular stratification. The important difference between toric varieties and stratified toric varieties is that the latter come with a stratification which may be coarser than the one given by orbits. As a consequence, the combinatorial object associated to a stratified toric variety, called a *semifan*, consists of those faces of the fan of the toric variety corresponding to strata. The faces which do not correspond to strata are ignored (see Definition 3.1.5).

By a *stratified toroidal variety* we mean a stratified variety (X, S) such that for any $x \in s$, $s \in S$ there is an étale map called a *chart* $\phi_x : U_x \to X_\sigma$ from an open neighborhood to a stratified toric variety such that all strata in U_x are preimages of strata in X_σ (see Definition 4.1.5). A toroidal embedding is a particular example of a stratified toroidal variety if we consider the stratification described in section 1.1 (see also Definition 4.1.8).

Each chart associates with a stratum s the *semicone* σ i.e the semifan consisting of faces of σ corresponding to strata in $\mathrm{Star}(s,S)$ for the corresponding stratum s.

If one chart associates to a stratum a semicone σ_1, and another associates σ_2 then Demushkin's theorem says that $\sigma_1 \simeq \sigma_2$.

This gives us the correspondence between strata and cones with semifans. Consequently, if $s \leq s'$, then σ is a face of σ' and the semicone σ is a subset of the semicone σ'. By glueing the semicones σ and σ' along the common faces σ'' we obtain the so called *semicomplex* Σ associated to the stratified toroidal variety. The semicomplex is determined uniquely up to isomorphism. Its faces of the semicomplex are indexed by the stratification. If a stratified toroidal variety is a toroidal embedding then the associated semicomplex is the usual conical complex.

Note that in contrast in the case of toroidal embeddings only some integral vectors v in faces of this semicomplex determine unique valuations on X which do not depend upon a chart. These vectors are called *stable* and the valuations they induce are also called *stable* (see Definitions 5.3.1, 5.3.2). The stable vectors form a subset in the support of the semicomplex, which we call the *stable support* (see Definition 6.1.1).

In particular, let X be a smooth surface and S be its stratification consisting of a point $p \in X$ and its complement. The associated semicomplex consists of the regular cone $\mathbf{Q}_{\geq 0} \cdot e_1 + \mathbf{Q}_{\geq 0} \cdot e_2$ corresponding to the point p and of the vertex corresponding to a big stratum. The only stable valuation is the valuation of the point p which corresponds to the unique (up to proportionality) stable vector $e_1 + e_2$.

The notion of stable support plays a key role in the theory of stratified toroidal varieties. The stable support has many nice properties. It intersects the relative interiors of all faces of the semicomplex and this intersection is polyhedral. The relative interiors of faces which are not in the semicomplex do not intersect the stable support. Consequently, all stable valuations are centered at the closures of strata. The part of the stable support which is contained in a face of the semicomplex is convex and is a union of relative interiors of a finite number of polyhedral cones. Also some "small" vectors like those which are in the parallelogram determined by ray generators belong to the stable support. If we consider the canonical resolution of toric singularities then all divisors occurring in this resolution determine stable valuations. In section 13, where we discuss another approach to the weak factorization conjecture in [4], we also show a method of computing the stable support which is based on the idea of torification of Abramovich and de Jong [2].

The stability condition for vectors in a face σ of the semicomplex Σ can be expressed in terms of invariance under the action of the group G^σ of automorphisms of a scheme $\widehat{X}_\sigma$ associated to this face (see Definition 5.3.1) (see section 4.13). The vector v is stable if the valuation determined by this vector on $\widetilde{X}_\sigma$ is G^σ-invariant. (Definition 5.3.2).

Birational toroidal mophisms or simply *toroidal morphisms* between stratified toroidal varieties are those induced locally by toric morphisms via toric charts and satisfying the Hironaka condition (see Definition 4.12.2). The Hironaka condition implies that toroidal morphisms, which are a priori induced by toric charts, in fact do not depend upon the charts and depend eventually only on the subdivisions of the semicomplex associated to the original stratified toroidal variety.

This leads to a $1-1$ correspondence between toroidal morphisms and certain subdivisions, called *canonical*, of the semicomplex associated to the original variety (see Theorems 4.14.1 and 6.6.1).

The canonical subdivisions are those satisfying the following condition: all "new" rays of the subdivision are in the stable support (see Proposition 7.6.1). It is not difficult to see the necessity of the condition since the "new" rays correspond to invariant divisors which determine invariant valuations. It is more complicated to show the sufficiency of the condition.

Let $(Y, R) \to (X, S)$ be a toroidal morphism of stratified toroidal varieties associated to the canonical subdivision Δ of the semicomplex Σ associated to (X, S). Then the semicomplex Σ_R associated to (Y, R) consists of all faces in Δ whose relative interiors intersect the stable support of Σ (see Theorem 6.6.1).

Toroidal morphisms between stratified toroidal varieties generalize those defined for toroidal embeddings. In that case, all subdivisions are canonical and we get the correspondence between toroidal embeddings and subdivisions of associated semicomplexes mentioned in Section 1.1 above.

In our considerations we require some condition, called orientability, of compatibility of charts on stratified toroidal varieties. This condition is analogous to the orientability of charts on an oriented differentiable manifold. It says that two étale charts $\phi : U \to X_{\sigma_s}$ and $\phi' : U \to X_{\sigma_s}$ define an automorphism $\widehat{\phi'} \circ \widehat{\phi}^{-1}$ in the identity component of the group of automorphisms of $\mathrm{Spec}(\widehat{\mathcal{O}}_{X_\sigma})$ (see Definition 4.9.1). In the differential setting, the group of local automorphisms consists of two components. In the algebraic situation the number of components depends upon the singularities. It is finite for toric singularities. Smooth toroidal varieties are oriented, since the group of automorphisms of the completion of a regular local ring is connected.

Let X be a toroidal variety with isolated singularity $x_1 \cdot x_2 = x_3 \cdot x_4$ or equivalently after some change of coordinates $y_1^2 + y_1^2 + y_3^2 + y_4^2 = 0$. In this case the group of automorphisms of the formal completion of X at the singular point consists of two components. Consequently, there are two orientations on X (for more details see Example 4.10.7).

Toroidal embeddings are oriented stratified toroidal varieties (see Example 4.10.6). Stratified toroidal varieties are orientable (see Section 8.2).

1.3. Toroidal varieties with torus action.
Let Γ denote any algebraic subgroup of an algebraic torus $T := K^* \times \ldots \times K^*$. If Γ acts on a stratified toric variety via a group homomorphism $\Gamma \to T$, then we require additionally, that the stabilizers at closed points in a fixed stratum s are all the same and equal to a group Γ_s associated to s, and there is a Γ_s-equivariant isomorphism of the local rings at these points.

A Γ-*stratified toroidal variety* is a stratified toroidal variety with a Γ-action which is locally Γ-equivariantly étale isomorphic to a Γ-stratified toric variety. We associate with it a Γ-semicomplex which is the semicomplex Σ with associated groups $\Gamma_\sigma \subset \Gamma$ acting on X_σ.

Γ-semicomplexes determine local projections $\pi : \sigma \to \sigma^\Gamma$ defined by good quotients $X_{\sigma_s} \to X_{\sigma_s^\Gamma} := X_\sigma / \Gamma_\sigma$. These local projections are coherent in the sense that they commute with face restrictions and subdivisions.

All the definitions in this category are adapted from the theory of stratified toroidal varieties. We additionally require that charts, morphisms and automorphisms occurring in the definitions are Γ_σ or Γ-equivariant.

1.4. **Birational cobordisms, Morse theory and polyhedral cobordisms of Morelli.** The theory of birational cobordisms, which was developed in [73], was inspired by Morelli's proof of the weak factorization theorem for toric varieties, where the notion of combinatorial cobordism was introduced (see [49]).

By a *birational cobordism* between birational varieties X and X' we understand a variety B with the action of the multiplicative group K^* such that the "lower boundary" B_- of B (resp. the "upper boundary" B_+) is an open subset which consists of orbits with no limit at 0 (resp. ∞). The cobordant varieties X and X' are isomorphic to the spaces of orbits (geometric quotients) B_-/K^* and B_+/K^* respectively. In differential cobordism the action of the 1-parameter group of diffeomorphisms $G \simeq (\mathbf{R}, +) \simeq (\mathbf{R}^*, \cdot)$, defined by the gradient field of a Morse function gives us an analogous interpretation of Morse theory. The bottom and top boundaries determined by a Morse function are isomorphic to the spaces of all orbits with no limit at $-\infty$ and $+\infty$ respectively (or 0 and $+\infty$ in multiplicative notation). The critical points of the Morse function are the fixed points of the action. "Passing through" these points induces simple birational transformations analogous to spherical transformations in differential geometry. If B is a smooth cobordism, then these birational transformations locally replace one weighted projective space with a complementary weighted projective space. These birational transformations can be nicely described using the language of toric varieties and associated fans. The semiinvariant parameters in the neighbouhood of a fixed point provide a chart which is a locally analytic isomorphism of the tangent spaces with the induced linear action. Hence locally, the cobordism is isomorphic to an affine space with a linear hyperbolic action. Locally we can identify B with an affine space $\mathbf{A}^n$, which is a toric variety corresponding to a regular cone σ in the vector space $N^{\mathbf{Q}}$. The action of K^* determines a 1-parameter subgroup, hence an integral vector $v \in N^{\mathbf{Q}}$. The sets B_- and B_+ correspond to the sets σ_+ and σ_- of all faces of σ visible from above or below (with respect to the direction of v), hence to the "upper" and "lower" boundaries of σ. The vector v defines a projection $\pi : N^{\mathbf{Q}} \to N^{\mathbf{Q}}/\mathbf{Q}v$. The quotient spaces B_-/K^*, B_+/K^* correspond to two "cobordant" (in the sense of Morelli) subdivisions $\pi(\sigma_+)$ and $\pi(\sigma_-)$ of the cone $\pi(\sigma)$. The problem lies in the fact that the fans $\pi(\sigma_+)$, $\pi(\sigma_-)$ are singular (which means that they are not spanned by a part of an integral basis). Consequently, the corresponding birational transformation is a composition of weighted blow-ups and blow-downs between singular varieties. The process of resolution of singularities of the quotient spaces is called π-desingularization and can be achieved locally by a combinatorial algorithm. The difficulty is now in patching these local π-desingularizations. This problem can be solved immediately once the theory of stratified toroidal varieties (with K^*-action) is established.

1.5. **Sketch of proof.** Let $B = B(X, X')$ be a smooth cobordism between smooth varieties X and X'. Let S be the stratification determined by the isotropy groups. Then (B, S) is a stratified toroidal variety with K^*-action. We can associate with it a K^*-semicomplex Σ. The K^*-semicomplex Σ determines for any face $\sigma_s \in \Sigma$ a projection $\pi_\sigma : \sigma \to \sigma^\Gamma$ determined by the good quotients $X_\sigma \to X_\sigma/\Gamma_\sigma = X_\sigma^\Gamma$. We apply a combinatorial algorithm - the so called π-desingularization Lemma of Morelli - to the semicomplex associated to the cobordism. The algorithm consists of starf subdivisions at stable vectors. Such subdivisions are canonical. As a result we obtain a semicomplex with the property that the projections of all simplices are either regular (nonsingular) cones or cones of smaller dimension (than the dimension of the projected cones) (see Lemma 10.4.4 for details). The semicomplex corresponds to a π-regular toroidal cobordism, all of whose open affine fixed point free subsets have smooth geometric quotients. The existence of such a cobordism easily implies the weak factorization theorem. (The blow-ups, blow-downs and flips induced by elementary cobordisms are regular (smooth)). Since each flip is a composition of a blow-up and a blow-down at a smooth center we come to a factorization of the map $X-\to X'$ into blow-ups and blow-downs at smooth centers (see Proposition 12.2.1).

2. PRELIMINARIES

2.1. **Basic notation and terminology.** Let $N \simeq \mathbf{Z}^k$ be a lattice contained in the vector space $N^{\mathbf{Q}} :=$ $N \otimes \mathbf{Q} \supset N$. By a *cone* in this paper we mean a convex set $\sigma = \mathbf{Q}_{\geq 0} \cdot v_1 + \ldots + \mathbf{Q}_{\geq 0} \cdot v_k \subset N^{\mathbf{Q}}$. By abuse of language we shall speak of a *cone σ in a lattice N*. To avoid confusion we shall sometimes write (σ, N) for the cone σ in N. For a cone σ in N denote by $N_\sigma := N \cap \mathrm{lin}(\sigma)$ the sublattice generated by σ. Then by

$$\underline{\sigma} := (\sigma, N_\sigma)$$

we denote the corresponding cone in N_σ. (Sometimes the cones σ and $\underline{\sigma}$ will be identified).

We call a vector $v \in N$ *primitive* if it generates sublattice $\mathbf{Q}_{\geq 0} v_i \cap N$. Each ray $\rho \in N^{\mathbf{Q}}$ contains a unique primitive ray $\mathrm{prim}(\rho)$. If $v_1, \ldots, v_k$ form a minimal set of primitive vectors generating $\sigma \subset N^{\mathbf{Q}}$ in the above sense, then we write

$$\sigma = \langle v_1, \ldots, v_k \rangle.$$

If σ contains no line we call it *strictly convex*. All cones considered in this paper are strictly convex. For any σ denote by $\mathrm{lin}(\sigma)$ the linear span of σ. For any cones σ_1 and σ_2 in N we write

$$\sigma = \sigma_1 + \sigma_2$$

if $\sigma = \{v_1 + v_2 \mid v_1 \in \sigma_1, v_2 \in \sigma_2\}$, and

$$\sigma = \sigma_1 \oplus \sigma_2$$

if $\mathrm{lin}(\sigma_1) \cap \mathrm{lin}(\sigma_2) = \{0\}$ and for any $v \in \sigma \cap N$ there exist $v_1 \in \sigma_1 \cap N$ and $v_2 \in \sigma_2 \cap N$ such that $v = v_1 + v_2$. For any cones σ_1 in N_1 and σ_2 in N_2 we define the cone $\sigma_1 \times \sigma_2$ in $N_1 \times N_2$ to be

$$\sigma_1 \times \sigma_2 := \{(v_1, v_2) \mid v_i \in \sigma_i \quad \text{for} \quad i = 1, 2\}.$$

We say that a cone σ in N is *regular* if there exist vectors $e_1, .., e_k \in N$ such that

$$\sigma = \langle e_1 \rangle \oplus \ldots \oplus \langle e_k \rangle.$$

A cone σ is *simplicial* if $\sigma = \langle v_1, \ldots, v_k \rangle$ is generated by linearly independent vectors. We call σ *indecomposable* if it cannot be represented as $\sigma = \sigma' \oplus \langle e \rangle$ for some nonzero vector $e \in N$.

Lemma 2.1.1. *Any cone σ in N is uniquely represented as*

$$\sigma = \mathrm{sing}(\sigma) \oplus \langle e_1, \ldots, e_k \rangle,$$

where $\langle e_1, \ldots, e_k \rangle$ is a regular cone and $\mathrm{sing}(\sigma)$ is the maximal indecomposable face of σ. □

By $\mathrm{int}(\sigma)$ we denote the relative interior of σ.

For any simplicial cone $\sigma = \langle v_1, \ldots, v_k \rangle$ in N set

$$\mathrm{par}(\sigma) := \{v \in \sigma \cap N_\sigma \mid v = \alpha_1 v_1 + \ldots + \alpha_k v_k, \text{where } 0 \leq \alpha_i < 1\},$$

$$\overline{\mathrm{par}(\sigma)} := \{v \in \sigma \cap N_\sigma \mid v = \alpha_1 v_1 + \ldots + \alpha_k v_k, \text{where } 0 \leq \alpha_i \leq 1\}.$$

For any simplicial cone $\sigma = \langle v_1, \ldots, v_k \rangle$ in N, by $\det(\sigma)$ we mean $\det(v_1, \ldots, v_k)$, where all vectors are considered in some basis of $N \cap \mathrm{lin}(\sigma)$ (a change of basis can only change the sign of the determinant).

Definition 2.1.2. A *minimal generator* of a cone σ is a vector not contained in a one dimensional face of σ and which cannot be represented as the sum of two nonzero integral vectors in σ. A *minimal internal vector* of a cone σ is a vector in $\mathrm{int}(\sigma)$ which cannot be represented as the sum of two nonzero integral vectors in σ such that at least one of them belongs to $\mathrm{int}(\sigma)$.

Immediately from the definition we get

Lemma 2.1.3. *Any minimal generator v of σ is a minimal internal vector of the face σ_v of σ, containing v in its relative interior.* □

Lemma 2.1.4. *For any simplicial σ each vector from $\mathrm{par}(\sigma)$ can be represented as a nonnegative integral combination of minimal generators.*

□

2.2. Toric varieties.

Definition 2.2.1. (see [20], [56]). By a *fan* Σ in N we mean a finite collection of finitely generated strictly convex cones σ in N such that

- any face of a cone in Σ belongs to Σ,
- any two cones of Σ intersect in a common face.

By the *support* of the fan we mean the union of all its faces, $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$.

If σ is a face of σ' we shall write $\sigma \preceq \sigma'$.

If $\sigma \preceq \sigma'$ but $\sigma \neq \sigma'$ we shall write $\sigma \prec \sigma'$.

For any set of cones Σ in N by $\overline{\Sigma}$ we denote the set $\{\tau \mid \tau \prec \tau' \quad \text{for some} \quad \tau' \in \Sigma\}$

Definition 2.2.2. Let Σ be a fan and $\tau \in \Sigma$. The *star* of the cone τ and the *closed star* of Σ are defined as follows:

$$\text{Star}(\tau, \Sigma) := \{\sigma \in \Sigma \mid \tau \preceq \sigma\},$$
$$\overline{\text{Star}}(\tau, \Sigma) := \{\sigma \in \Sigma \mid \sigma' \preceq \sigma \text{ for some } \sigma' \in \text{Star}(\tau, \Sigma)\},$$

Definition 2.2.3. Let Σ_i be a fan in N_i for $i = 1, 2$. Then the *product* of Σ_1 and Σ_2 is a fan $\Sigma_1 \times \Sigma_2$ in $N_1^{\mathbf{Q}} \times N_2^{\mathbf{Q}}$ defined as follows:

$$\Sigma_1 \times \Sigma_2 := \{\sigma_1 \times \sigma_2 \mid \sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2\}.$$

To a fan Σ there is associated a toric variety $X_\Sigma \supset T$, i.e. a normal variety on which a torus T acts effectively with an open dense orbit (see [39], [21], [56], [25]). To each cone $\sigma \in \Sigma$ corresponds an open affine invariant subset X_σ and its unique closed orbit O_σ. The orbits in the closure of the orbit O_σ correspond to the cones of $\text{Star}(\sigma, \Sigma)$.

Denote by

$$M := \text{Hom}_{alg.gr.}(T, K^*)$$

the lattice of group homomorphisms to K^*, i.e. characters of T. Then the dual lattice $N = \text{Hom}_{alg.gr.}(K^*, T)$ can be identified with the lattice of 1-parameter subgroups of T. Then vector space $M^{\mathbf{Q}} := M \otimes \mathbf{Q}$ is dual to $N^{\mathbf{Q}} := N \otimes \mathbf{Q}$. Let (v, w) denote the relevant pairing for $v \in N, w \in M$

For any $\sigma \subset N^{\mathbf{Q}}$ we denote by

$$\sigma^\vee := \{m \in M \mid (v, m) \geq 0 \text{ for any } m \in \sigma\}$$

the set of integral vectors of the dual cone to σ. Then the ring of regular functions $K[X_\sigma]$ is $K[\sigma^\vee]$.

Each vector $v \in N$ defines a linear function on M which determines a valuation $\text{val}(v)$ on X_Σ.

For any regular function $f = \sum_{w \in M} a_w x^w \in K[T]$ set

$$\text{val}(v)(f) := \min\{(v, w) \mid a_w \neq 0\}.$$

Thus N can be perceived as the lattice of all T-invariant integral valuations of the function field of X_Σ.

For any $\sigma \subset N^{\mathbf{Q}}$ set

$$\sigma^\perp := \{m \in M \mid (v, m) = 0 \text{ for any } m \in \sigma\}.$$

The latter set represents all invertible characters on X_σ. All noninvertible characters are 0 on O_σ. The ring of regular functions on $O_\sigma \subset X_\sigma$ can be written as $K[O_\sigma] = K[\sigma^\perp] \subset K[\sigma^\vee] = K[\sigma^\perp] \otimes K[\underline{\sigma}]$. Thus $K[X_\sigma] = K[O_\sigma][\underline{\sigma}^\vee]$

Let $T_\sigma \subset T$ be the subtorus corresponding the sublattice $N_\sigma := \text{lin}(\sigma) \cap N$ of N. Then by definition, T_σ acts trivially on $K[O_\sigma] = K[\sigma^\perp] = K[X_\sigma]^{T_\sigma}$. Thus $T_\sigma = \{t \in T \mid tx = x, x \in O_\sigma\}$.

This leads us to the lemma

Lemma 2.2.4. *Any toric variety X_σ is isomorphic to $X_{\underline{\sigma}} \times O_\sigma$, where $O_\sigma \simeq T/T_\sigma$.* $\qquad \square$

For any $\sigma \in \Sigma$ the closure $\overline{O}_\sigma$ of the orbit $O_\sigma \subset X_\Sigma$ is a toric variety with the big torus T/T_σ. Let $\pi : N \to N/N_\sigma$ be the natural projection. Then $\overline{O}_\sigma$ corresponds to the fan $\Sigma' := \{\frac{\tau + \text{lin}(\sigma)}{\text{lin}(\sigma)} \mid \tau \in \text{Star}(\tau, \Sigma)\}$ in N/N_σ.

2.3. **Morphisms of toric varieties.**

Definition 2.3.1. (see [39], [56], [21], [25]). A *birational toric morphism* or simply a *toric morphism* of toric varieties $X_\Sigma \to X_{\Sigma'}$ is a morphism identical on $T \subset X_\Sigma, X_{\Sigma'}$

Definition 2.3.2. (see [39], [56], [21], [25]). A *subdivision* of a fan Σ is a fan Δ such that $|\Delta| = |\Sigma|$ and any cone $\sigma \in \Sigma$ is a union of cones $\delta \in \Delta$.

Definition 2.3.3. Let Σ be a fan and ϱ be a ray passing in the relative interior of $\tau \in \Sigma$. Then the *star subdivision* $\varrho \cdot \Sigma$ of Σ with respect to ϱ is defined to be

$$\varrho \cdot \Sigma = (\Sigma \setminus \operatorname{Star}(\tau, \Sigma)) \cup \{\varrho + \sigma \mid \sigma \in \overline{\operatorname{Star}}(\tau, \Sigma) \setminus \operatorname{Star}(\tau, \Sigma)\}.$$

If Σ is regular, i.e. all its cones are regular, $\tau = \langle v_1, \dots, v_l \rangle$ and $\varrho = \langle v_1 + \dots + v_l \rangle$ then we call the star subdivision $\varrho \cdot \Sigma$ *regular*.

Proposition 2.3.4. *(see [39], [21], [56], [25]). Let X_Σ be a toric variety. There is a 1-1 correspondence between subdivisions of the fan Σ and proper toric morphisms $X_{\Sigma'} \to X_\Sigma$.* □

Remark. Regular star subdivisions from 2.3.3 correspond to blow-ups of smooth varieties at closures of orbits ([56], [25]). Arbitrary star subdivisions correspond to blow-ups of some ideals associated to valuations (see Lemma 5.2.8).

2.4. **Toric varieties with Γ-action.** In further considerations let Γ denote any algebraic subgroup of an algebraic torus $T = K^* \times \dots \times K^*$. By X/Γ and $X//\Gamma$ we denote respectively geometric and good quotients. If Γ acts on an algebraic variety X and $x \in X$ is a closed point, then Γ_x denotes the isotropy group of x.

Definition 2.4.1. Let Γ act on $X_\sigma \supset T$ via a group homomomorphism $\Gamma \to T$. Set
$$\Gamma_\sigma := \{g \in \Gamma \mid g(x) = x \quad \text{for any} \quad x \in O_\sigma\}.$$
We shall write
$$\sigma = \sigma' \oplus^\Gamma \langle e_1, \dots, e_k \rangle$$
if $\sigma = \sigma' \oplus \langle e_1, \dots, e_k \rangle$, $\langle e_1, \dots, e_k \rangle$ is regular and $\Gamma_\sigma = \Gamma_{\sigma'}$. We say that σ is Γ-*indecomposable* if it cannot be represented as $\sigma = \sigma' \oplus^\Gamma \langle e_1, \dots, e_k \rangle$ (or equivalently X_σ is not of the form $X_{\sigma'} \times \mathbf{A}^k$ with Γ_σ acting trivially on $\mathbf{A}^k$). By
$$\operatorname{sing}^\Gamma(\sigma)$$
we mean the maximal Γ-indecomposable face of σ.

If Γ acts on a toric vatiety via a group homomomorphism $\Gamma \to T$ we shall speak of a *toric action* of Γ.

2.5. **Demushkin's Theorem.** For any algebraic variety X and its (in general nonclosed) point $x \in X$ we denote by $\widehat{\mathcal{O}}_{X,x}$ the completion of the local ring $\mathcal{O}_{X,x}$ at the maximal ideal of x. We also set
$$\widehat{X}_x := \operatorname{Spec}(\widehat{\mathcal{O}}_{X,x}).$$
For the affine toric variety X_σ define
$$\widehat{X}_\sigma := \operatorname{Spec}(\widehat{\mathcal{O}}_{X_\sigma, O_\sigma}).$$
We shall use the following Theorem of Demushkin ([22]):

Theorem 2.5.1. *Let σ and τ be two cones of maximal dimension in isomorphic lattices $N_\sigma \simeq N_\tau$.*

Then the following conditions are equivalent:

1. $\sigma \simeq \tau$.
2. $\widehat{X}_\sigma \simeq \widehat{X}_\tau$.

Proof. For the proof see [22] or the proof of 4.6.1. □

The above theorem can be formulated for affine toric varieties with Γ-action.

Theorem 2.5.2. *Let σ and τ be two cones of maximal dimension in isomorphic lattices $N_\sigma \simeq N_\tau$. Let Γ act on $X_\sigma \supset T_\sigma$ and on $X_\tau \supset T_\tau$ via group homomorpisms $\Gamma \to T_\sigma$ and $\Gamma \to T_\tau$.*

Then the following conditions are equivalent:

1. *There exists an isomorphism of cones $\sigma \simeq \tau$ inducing a Γ-equivariant isomorphism of toric varieties.*
2. *There exists a Γ-equivariant isomorphism $\widehat{X}_\sigma \simeq \widehat{X}_\tau$.*

Proof. For the proof see the proof of 4.6.1 or [22] . $\qquad\square$

Definition 2.5.3. Let Γ act on $X_\sigma \supset T_\sigma$ and on $X_\tau \supset T_\tau$ via group homomorpisms $\Gamma \to T_\sigma$ and $\Gamma \to T_\tau$. We say that cones σ and τ are Γ-*isomorphic* if there exists an isomorphism of cones $\sigma \simeq \tau$ inducing a Γ-equivariant isomorphism of toric varieties.

The above theorem can be formulated as follows:

Theorem 2.5.4. *Let σ and τ be two cones in isomorphic lattices $N_\sigma \simeq N_\tau$. Let Γ act on $X_\sigma \supset T_\sigma$ and on $X_\tau \supset T_\tau$ via group homomorpisms $\Gamma \to T_\sigma$ and $\Gamma \to T_\tau$. Assume that $\Gamma_\sigma = \Gamma_\tau$.*

Then the following conditions are equivalent:

1. $\mathrm{sing}^\Gamma(\sigma)$ *and* $\mathrm{sing}^\Gamma(\tau)$ *are* Γ_σ-*isomorphic.*
2. *For any closed points* $x_\sigma \in O_\sigma$ *and* $x_\tau \in O_\tau$, *there exists a* Γ_σ-*equivariant isomorphism of the local rings* $\mathcal{O}_{X_\sigma,x_\sigma}$ *and* $\mathcal{O}_{X_\tau,x_\tau}$.
3. *For any closed points* $x_\sigma \in O_\sigma$ *and* $x_\tau \in O_\tau$, *there exists a* Γ_σ-*equivariant isomorphism of the completions of the local rings* $\widehat{\mathcal{O}}_{X_\sigma,x_\sigma}$ *and* $\widehat{\mathcal{O}}_{X_\tau,x_\tau}$.

Proof. $(1) \Rightarrow (2)$ Write $\sigma = \mathrm{sing}^\Gamma(\sigma) \oplus^\Gamma \mathrm{r}(\sigma)$ and $\tau = \mathrm{sing}^\Gamma(\tau) \oplus^\Gamma \mathrm{r}(\tau)$, where $\mathrm{r}(\sigma)$, $\mathrm{r}(\tau)$ denote regular cones. By Lemma 2.2.4, $X_\sigma = X_{\underline{\sigma}} \times O_\sigma = X_{\underline{\mathrm{sing}^\Gamma(\sigma)}} \times X_{\mathrm{r}(\sigma)} \times O_\sigma$, where Γ_σ acts trivially on $X_{\mathrm{r}(\sigma)} \times O_\sigma$. Thus there exists a Γ_σ-equivariant isomorphism $\mathcal{O}_{X_\sigma,x_\sigma} \simeq \mathcal{O}_{X_{\underline{\mathrm{sing}^\Gamma(\sigma)} \times X_{\mathrm{r}'(\sigma)}}}$, where $\mathrm{r}'(\sigma)$ is a regular cone of dimension $\dim(N_\sigma) - \dim(\mathrm{sing}^\Gamma(\sigma))$ and Γ_σ acts trivially on $X_{\mathrm{r}'(\sigma)}$. Analogously $\mathcal{O}_{X_\tau,x_\tau} \simeq \mathcal{O}_{X_{\underline{\mathrm{sing}^\Gamma(\tau)} \times X_{\mathrm{r}'(\tau)}}} \simeq \mathcal{O}_{X_{\underline{\mathrm{sing}^\Gamma(\sigma)} \times X_{\mathrm{r}'(\sigma)}}} = \mathcal{O}_{X_\sigma,x_\sigma}$.

The implication $(2) \Rightarrow (3)$ is trivial. For the proof of $(3) \Rightarrow (1)$ we find a regular cone $\mathrm{re}(\sigma)$ in N_σ such that $\sigma' := \sigma \oplus \mathrm{re}(\sigma)$ is of maximal dimension in N_σ. Since Γ_σ acts trivially on O_σ and on O_τ we have $\sigma' := \sigma \oplus^{\Gamma_\sigma} \mathrm{re}(\sigma)$. Analogously $\tau' := \tau \oplus^{\Gamma_\tau} \mathrm{re}(\tau)$. By Theorem 2.5.2, σ' and τ' are Γ_σ-isomorphic. Consequently, $\mathrm{sing}^\Gamma(\sigma) = \mathrm{sing}^{\Gamma_\sigma}(\sigma') \simeq \mathrm{sing}^{\Gamma_\tau}(\tau') = \mathrm{sing}^\Gamma(\tau)$. $\qquad\square$

Theorem 2.5.4 allows us to assign a singularity type to any closed point x of a toric variety X with a toric action of the group Γ:

Definition 2.5.5. By the *singularity type* of a point x of a toric variety X we mean the function

$$\mathrm{sing}(x) := \underline{\mathrm{sing}(\sigma_x)},$$

where σ_x is a cone of maximal dimension such that $\widehat{X}_x \simeq \widehat{X}_{\sigma_x}$.

By the *singularity type* of a point x of a toric variety X with a toric action of group Γ we mean

$$\mathrm{sing}^\Gamma(x) := (\Gamma_x, \underline{\mathrm{sing}^{\Gamma_x}(\sigma_x)}),$$

where σ_x is a cone of maximal dimension with toric action of Γ_x on $\widehat{X}_{\sigma_x}$ and such that there exists a Γ_x-equivariant isomorphism $\widehat{X}_x \simeq \widehat{X}_{\sigma_x}$.

3. Stratified toric varieties and semifans

3.1. **Definition of a stratified toric variety.** In this section, we give a combinatorial description of stratified toric varieties in terms of so called (embedded) semifans.

Definition 3.1.1. Let X be a noetherian scheme X over $\mathrm{Spec}(K)$. A *stratification* of X is a decomposition of X into a finite collection S of pairwise disjoint locally closed irreducible smooth subschemes $s \subset X$, called strata, with the following property: For every $s \in S$, the closure $\overline{s} \subset X$ is a union of strata.

Definition 3.1.2. Let S and S' be two stratifications of X. We say that S is *finer* than S' if any stratum in S' is a union of strata in S. In this case we shall also call S' *coarser* than S.

Definition 3.1.3. Let X be a toric variety with big torus $T \subset X$. A *toric stratification* of X is a stratification S of X consisting of T-invariant strata s such that for any two closed points $x, x' \in s$ their local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,x'}$ are isomorphic.

Definition 3.1.4. Let X be a toric variety with a toric action of Γ. We say that a toric stratification S of X is *compatible with the action of* Γ if

1. All points in the same stratum s have the same isotropy group Γ_s.
2. For any two closed points x, y from one stratum s there exists a Γ_s-equivariant isomorphism $\alpha : \widehat{X}_x \to \widehat{X}_y$, preserving all strata.

If S is a toric stratification of X, then we shall also speak of a *stratified toric variety* (X, S). If X is a toric variety with a toric stratification , compatible with Γ, then we shall also speak of a Γ-*stratified toric variety* (X, S).

The combinatorial objects we shall use in this context are the following:

Definition 3.1.5. An *embedded semifan* is a subset $\Omega \subset \Sigma$ of a fan Σ in a lattice N such that for every $\sigma \in \Sigma$ there is an $\omega(\sigma) \in \Omega$ satisfying

1. $\omega(\sigma) \preceq \sigma$ and any other $\omega \in \Omega$ with $\omega \preceq \sigma$ is a face of $\omega(\sigma)$,
2. $\sigma = \omega(\sigma) \oplus \mathrm{r}(\sigma)$ for some regular cone $\mathrm{r}(\sigma) \in \Sigma$.

A *semifan* in a lattice N is a set Ω of cones in N such that the set Σ of all faces of the cones of Ω is a fan in N and $\Omega \subset \Sigma$ is an embedded semifan.

Definition 3.1.6. An embedded semifan $\Omega \subset \Sigma$ with a toric action of Γ on X_Σ will be called an *embedded* Γ-*semifan* if for every $\sigma \in \Sigma$, $\sigma = \omega(\sigma) \oplus^\Gamma \mathrm{r}(\sigma)$.

Remark. A semifan or an embedded semifan can be viewed as a Γ-semifan or an embedded Γ-semifan with trivial group Γ.

Some examples are discussed at the end of this section. The main statement of this section says that stratified toric varieties are described by embedded semifans:

Proposition 3.1.7. *Let Σ be a fan in a lattice N, and let X denote the associated toric variety with a toric action of Γ. There is a canonical 1-1 correspondence between the toric stratifications of X compatible withn the action of Γ and the embedded Γ-semifans $\Omega \subset \Sigma$:*

1. *If S is a toric stratification of X, compatible with the action of Γ, then the corresponding embedded Γ-semifan $\Omega \subset \Sigma$ consists of all those cones $\omega \in \Sigma$ that describe the big orbit of some stratum $s \in S$.*
2. *If $\Omega \subset \Sigma$ is an embedded Γ-semifan, then the strata of the associated toric stratification S of X arise from the cones of Ω via*

$$\omega \mapsto \mathrm{strat}(\omega) := \bigcup_{\omega(\sigma)=\omega} O_\sigma.$$

Proof. $(1) \Rightarrow (2)$ Since strata of S are T-invariant and disjoint, each orbit O_τ belongs to a unique stratum s. Let $\omega \in \Omega$ describe the big open orbit of s. Then O_τ is contained in the closure of O_ω. Hence ω is a face of τ. By Definition de: embedded, $\Gamma_\tau = \Gamma_\omega$ and there exists a Γ_τ-equivariant isomorphism of the local rings $\mathcal{O}_{X_\tau,x}$ and $\mathcal{O}_{X_\omega,y}$ of two points $x \in O_\tau$ and $y \in O_\omega$. By Theorem 2.5.4 and since $\mathrm{sing}^\Gamma(\tau) \supset \mathrm{sing}^\Gamma(\omega)$, we infer that $\mathrm{sing}^\Gamma(\tau) = \mathrm{sing}^\Gamma(\omega)$. Hence $\omega = \mathrm{sing}^\Gamma(\omega) \oplus^\Gamma \mathrm{r}(\omega)$ and $\tau = \mathrm{sing}^\Gamma(\tau) \oplus^\Gamma \mathrm{r}(\tau) = \omega \oplus^\Gamma \mathrm{r}'(\tau)$, where $\mathrm{r}(\tau) = \mathrm{r}(\omega) \oplus \mathrm{r}'(\tau)$.

$(2) \Rightarrow (1)$. By definition $\{\sigma \in \Sigma \mid \omega(\sigma) = \omega\} = \mathrm{Star}(\omega, \Sigma) \setminus \bigcup_{\omega \prec \omega' \in \Omega} \mathrm{Star}(\omega', \Sigma)$. Hence all the defined subsets $\mathrm{strat}(\omega)$ are locally closed. The closure of each subset $\mathrm{strat}(\omega)$ corresponds to $\mathrm{Star}(\omega, \Sigma)$ and hence it is a union of the sets $\mathrm{strat}(\omega')$, where $\omega \prec \omega'$. Since $\tau = \omega \oplus^\Gamma \mathrm{r}(\tau)$ we have $\Gamma_\tau = \Gamma_\omega$. Each subset $\mathrm{strat}(\omega)$ is a toric variety with a fan $\Sigma' := \{\frac{\tau+\mathrm{lin}(\omega)}{\mathrm{lin}(\omega)} | \omega(\tau) = \omega\}$ in $(N^Q)' := N^Q / \mathrm{lin}(\omega)$. Since $\tau = \omega \oplus^\Gamma \mathrm{r}(\tau)$ the cone $\frac{\tau+\mathrm{lin}(\omega)}{\mathrm{lin}(\omega)}$ is isomorphic to the regular cone $\mathrm{r}(\tau)$ in $(N^Q)'$. Thus the strata $\mathrm{strat}(\omega)$ is smooth. The local rings of closed points of the the strata $\mathrm{strat}(\omega)$ have the same isotropy group Γ_ω. Moreover if $O_\tau \subset \mathrm{strat}(\omega)$ then $\mathrm{sing}^\Gamma(\tau) = \mathrm{sing}^\Gamma(\omega)$. By Theorem 2.5.4, we conclude that there exists a Γ-equivariant isomorphism of the local rings of any two points $x \in O_\tau$ and $y \in O_\omega$. $\qquad \square$

As a corollary from the above we obtain the following lemmas:

Lemma 3.1.8. $\mathrm{strat}(\omega) = \overline{\mathrm{strat}(\omega)} \setminus \bigcup_{\omega' \prec \omega} \overline{\mathrm{strat}(\omega')}$. $\qquad \square$

Lemma 3.1.9. *Let S_1 and S_2 be two toric stratifications on a toric variety X corresponding to two embedded semifans $\Omega_1, \Omega_2 \subset \Sigma$. Then S_1 is coarser than S_2 iff $\Omega_1 \subset \Omega_2$.*

Proof ($\Rightarrow$) Assume that S_1 is coarser than S_2. Let $\omega \in \Omega_1$ be the cone corresponding to a stratum $s_1 \in S_1$. The closure $\overline{s_1}$ is a union of strata in S_2. There is a generic stratum $s_2 \in S_2$ contained in $\overline{s_1}$. Then $\overline{s_1} = \overline{s_2}$ and $\omega \in \Omega_2$ corresponds to the stratum s_2.

($\Leftarrow$) Now assume that $\Omega_1 \subset \Omega_2$. Then the closure of any stratum s_1 equals $\overline{s_1} = \overline{\omega_1}$, where $\omega_1 \in \Omega_1$ is the cone corresponding to s_1. But then $\overline{\omega_1} = \overline{s_2}$, where $s_2 \in S_2$ is the stratum corresponding to $\omega_1 \in \Omega_2$. Thus $\overline{s_1} = \overline{s_2}$ and it suffices to apply Lemma 3.1.8. $\qquad\square$

Remark. Let (X, S) be the stratified toric variety arising from a semifan $\Omega \subset \Sigma$. The above correspondence between the cones of Ω and the strata of S is order reversing in the sense that $\omega \preceq \omega'$ iff $\mathrm{strat}(\omega')$ is contained in the closure of $\mathrm{strat}(\omega)$.

Lemma 3.1.10. *Let $\Omega \subset \Sigma$ be an embedded Γ-semifan. Then all Γ-indecomposable faces of Σ are in Ω.*

Proof. For a Γ-indecomposable face σ write $\sigma = \omega(\sigma) \oplus^\Gamma \mathrm{r}(\sigma)$. This implies $\sigma = \omega(\sigma) \in \Omega$. $\qquad\square$

We conclude the section with some examples. The first two examples show that for any fan there are a maximal and a minimal embedded semifan.

Example 3.1.11. If Σ is a fan in a lattice N, then $\Sigma \subset \Sigma$ is an embedded semifan. The corresponding stratification of the toric variety X associated to Σ is the decomposition of X into the orbits of the big torus $T \subset X$. The orbit stratification is the finest among all toric stratifications of X.

Example 3.1.12. For a fan Σ in a lattice N and a toric action of Γ on X_Σ let $\mathrm{Sing}^\Gamma(\Sigma)$ denote the set of all maximal indecomposable parts $\mathrm{sing}^\Gamma(\sigma)$, where $\sigma \in \Sigma$. Then $\mathrm{Sing}^\Gamma(\Sigma) \subset \Sigma$ is an embedded Γ-semifan, and for any other embedded Γ-semifan $\Omega \subset \Sigma$ one has $\mathrm{Sing}^\Gamma(\Sigma) \subset \Omega$. The toric stratification corresponding to $\mathrm{Sing}^\Gamma(\Sigma) \subset \Sigma$ is the stratification by singularity type on the toric variety X_Σ with the toric action of Γ. The toric stratification corresponding to $\mathrm{Sing}(\Sigma) \subset \Sigma$ is the coarsest among all toric stratifications of X. The toric stratification corresponding to $\mathrm{Sing}^\Gamma(\Sigma) \subset \Sigma$ is the coarsest among all toric stratifications of X which are compatible with the action of Γ.

Example 3.1.13. Let $\mathbf{A}^2 \supset K^* \times K^*$ be a toric variety with the strata $s_0 := \{(0,0)\}$, $s_1 := K^* \times \{0\}$, and $s_2 := \mathbf{A}^2 \setminus (s_0 \cup s_1)$. Then $\mathbf{A}^2$ with the above stratification is a stratified toric variety.

We have $\sigma_0 = \langle (1,0), (0,1) \rangle$, $\sigma_1 = \langle (0,1) \rangle$, $\sigma_2 = \{(0,0)\}$,
$\overline{\sigma_0} = \{\langle (1,0), (0,1) \rangle\}$, $\overline{\sigma_1} = \{\langle (0,1) \rangle\}$, $\overline{\sigma_2} = \{(0,0)\}, \langle (1,0) \rangle\}$.

4. Stratified toroidal varieties

4.1. Definition of a stratified toroidal variety.

Definition 4.1.1. (see also [20]). Let X be a variety or a noetherian scheme over K. X is called *toroidal* if for any $x \in X$ there exists an open affine neighborhood U_x and an étale morphism $\phi_x : U_x \to X_{\sigma_x}$ into some affine toric variety X_{σ_x}.

Definition 4.1.2. Let Γ act on noetherian schemes X and Y. We say that a Γ-equivariant morphism $f : Y \to X$ is Γ-*smooth* (resp. Γ-*étale*) if there is a smooth (resp. étale) morphism $f' : Y' \to X'$ of varieties with trivial action of Γ and Γ-equivariant fiber square

$$
\begin{array}{ccccc}
X \times_{X'} Y' & \simeq & Y & \xrightarrow{f} & X \\
& & \downarrow & & \downarrow \\
& & Y' & \xrightarrow{f'} & X'
\end{array}
$$

Lemma 4.1.3. *Let $Y \to X$ be a Γ-smooth morphism. If the good quotient $X//\Gamma$ exists then $Y//\Gamma$ exists and $Y = Y//\Gamma \times_{X//\Gamma} X$.*

Proof. Let $Y_0 := Y' \times_X X//\Gamma$. Then $Y = Y_0 \times_{X//\Gamma} X$ and $Y//\Gamma = Y_0$. $\qquad\square$

Definition 4.1.4. Let X be a toroidal variety or a toroidal scheme. We say that an action of a group Γ on X is *toroidal* if for any $x \in X$ there exists an open Γ-invariant neighborhood U and a Γ-equivariant, Γ_x-smooth, morphism $U \to X_\sigma$, into a toric variety X_σ with a toric action of $\Gamma \supset \Gamma_x$.

Definition 4.1.5. Let X be a toroidal variety or a toroidal scheme. We say that a stratification S of X is *toroidal* if any point x in a stratum $s \in S$ admits an open neighborhood U_x and an étale morphism $\phi_x : U_x \to X_\sigma$ into a stratified toric variety such that $s \cap U_x$ equals $\phi_x^{-1}(O_\sigma)$ and the intersections $s' \cap U$, $s' \in S$, are precisely the inverse images of strata in X_σ.

Definition 4.1.6. Let X be toroidal variety or a toroidal scheme with a toroidal action of Γ. A toroidal stratification S of X is said to be *compatible with the action of* Γ if

1. Strata of S are invariant with respect to the action of Γ.
2. All points in one stratum s have the same isotropy group Γ_s.
3. For any point $x \in s$ there exists a Γ-invariant neighborhood U and a Γ-equivariant, Γ_s-smooth morphism $U \to X_\sigma$ into a Γ-stratified toric variety, such that $s \cap U_x$ equals $\phi_x^{-1}(O_\sigma)$ and the intersections $s' \cap U$, $s' \in S$, are precisely the inverse images of strata in X_σ.

If X is a toroidal variety or a toroidal scheme with a toroidal stratification S then we shall speak of a *stratified toroidal variety* (resp. a *stratified toroidal scheme*). If X is a toroidal variety or a toroidal scheme with a toroidal action of Γ and a toroidal stratification S compatible with the action of Γ then we shall speak of a Γ-*stratified toroidal variety* (resp. Γ-*stratified toroidal scheme*).

If (X, S) is a Γ-stratified toroidal variety then for any stratum $s \in S$ set $\Gamma_s := \Gamma_x$, where $x \in s$. Note that if $s \leq s'$ then $\Gamma_s \subseteq \Gamma_{s'}$.

Remark. A stratified toroidal variety can be considered as a Γ-stratified toroidal variety with trivial action of the trivial group $\Gamma = \{e\}$.

A simple generalization of Example 3.1.12 is the following lemma:

Lemma 4.1.7. *Let X be a toroidal variety with a toroidal action of Γ. Let $\mathrm{Sing}^\Gamma(X)$ be the stratification determined by the singularity type $\mathrm{sing}^\Gamma(x)$ (see Definition 2.5.5) . Then $\mathrm{Sing}^\Gamma(X)$ is a toroidal stratification compatible with the action of Γ.* $\qquad\blacksquare$

Definition 4.1.8. (see also [39]). A *toroidal embedding* is a stratified toroidal variety such that any point p in a stratum $s \in S$ admits an open neighborhood U_p and an étale morphism of stratified varieties $\phi_p : U_p \to X_\sigma$ into a toric variety with orbit stratification (see Example 3.1.11).

A *toroidal embedding* with Γ-action is a Γ-stratified toroidal variety which is a toroidal embedding.

Remark. The definition of a toroidal embedding differs from the original definition in [39]. It is equivalent to the definition of a toroidal embedding without self-intersections (see Section 1.1).

4.2. **Existence of invariant stratifications on smooth varieties with Γ-action.**

Lemma 4.2.1.   1. *Let X be a smooth variety with Γ-action. Then there exists a toroidal stratification $\mathrm{Sing}^\Gamma(X)$ of X which is compatible with the action of Γ.*
2. *Let X be a smooth variety with Γ-action and D be a Γ-invariant divisor with simple normal crossings. Let S_D be the stratification determined by the components of the divisor D. Set*
$$\mathrm{sing}^{\Gamma,D}(x) := (\mathrm{sing}^\Gamma(x), D(x)),$$
where $D(x)$ is the set of components of D passing through $x \in X$

Then $\mathrm{sing}^{\Gamma,D}$ determines a toroidal stratification $\mathrm{Sing}^\Gamma(X, D)$ compatible with the action of Γ. In particular all the closures of strata from $\mathrm{Sing}^\Gamma(X, D)$ have normal crossings with components of D.

Proof. For any x we can find Γ-semiinvariant parameters $u_1, \ldots, u_k$ describing the orbit $\Gamma \cdot x$ in some Γ-invarinat neighborhood U_x of x.

Additionally in (2) we require that each component of D through $x \in X$ is described by one of the parameters in U_x. We define a smooth morphism $\phi_x : U_x \to \mathbf{A}^k$ by $\phi_x(y) = (u_1(y), \ldots, u_k(y))$. The action of Γ on X defines the action of Γ on $\mathbf{A}^k$ with standard coordinates be Γ-semiinvariant with corresponding

weights. Then $\mathrm{Sing}^\Gamma(\mathbf{A}^k, D_A)$, where $D_A = \overline{\phi(D)}$, defines a Γ-invariant stratification on $\mathbf{A}^k$, whose strata are determined by standard coordinates. Therefore $\mathrm{Sing}^\Gamma(X, D)$ is also a stratification on X which is locally a pull-back of $\mathrm{Sing}^\Gamma(\mathbf{A}^k, D_A)$.

The morphism ϕ_x is Γ-equivariant. Denote by $u_1, \ldots, u_k, \ldots, u_n$ all local Γ_x-semiinvariant parameters at x. Let $\widetilde{\phi}_x : U_x \to \mathbf{A}^n$ be the étale Γ_x-equivariant morphism defined by $\widetilde{\phi}_x(y) := (u_1(y), \ldots, u_n(y))$. By Luna's fundamental lemma (see Luna [42]) we can find a Γ_x-invariant neighborhood $U'_x \subset U_x$ for which the induced quotient morphism $\widetilde{\phi}_x/\Gamma_x : U'_x//\Gamma_x \to \mathbf{A}^n//\Gamma_x$ is étale and $U'_x \simeq U'_x//\Gamma_x \times_{\mathbf{A}^n//\Gamma_x} \mathbf{A}^n$. By Sumihiro [67] we can find a Γ-invariant open affine $V \subset \Gamma \cdot U'$ such that $V \simeq V//\Gamma_x \times_{V//\Gamma_x} \mathbf{A}^n$. Let $p : \mathbf{A}^n \to \mathbf{A}^k$ denote the standard projection on the first coordinates. Then $p\widetilde{\phi}_x : V \to \mathbf{A}^k$ is a Γ-equivariant Γ_x-smooth morphism. By definition $(X, \mathrm{Sing}^\Gamma(X, D))$ is a Γ-stratified toroidal variety. $\qquad\square$

4.3. **Conical semicomplexes.** Here we generalize the notion of a semifan. For this we first have to figure out those semifans that describe affine stratified toric varieties. In analogy to usual cones and fans we shall denote them by small Greek letters σ, τ, etc.:

Definition. Let N be a lattice. A *semicone* in N is a semifan σ in N such that the support $|\sigma|$ of σ occurs as an element of σ.

The *dimension* of a semicone is the dimension of its support. Moreover, for an injection $\imath : N \to N'$ of lattices, the *image* $\imath(\sigma)$ of a semicone σ in N is the semicone consisting of the images of all the elements of σ.

Note that every cone becomes a semicone by replacing it with the set of all its faces. Moreover, every semifan is a union of maximal semicones. Generalizing this observation, we build up semicomplexes from semicones:

Definition 4.3.1. Let Σ be a finite collection of semicones σ in lattices N_σ with $\dim(\sigma) = \dim(N_\sigma)$. Moreover, suppose that there is a partial ordering "$\leq$" on Σ.

We call Σ a *semicomplex* if for any pair $\tau \leq \sigma$ in Σ there is an associated linear injection $\imath_\tau^\sigma : N_\tau \to N_\sigma$ such that $\imath_\tau^\sigma(N_\tau) \subset N_\sigma$ is a saturated sublattice and

1. $\imath_\tau^\sigma \circ \imath_\varrho^\tau(\varrho) = \imath_\varrho^\sigma(\varrho)$,
2. $\imath_\varrho^\sigma(|\varrho|) = \imath_\tau^\sigma(|\tau|)$ implies $\varrho = \tau$,
3. $\sigma = \bigcup_{\tau \leq \sigma} \imath_\tau^\sigma(|\tau|) = \bigcup_{\tau \leq \sigma} \imath_\tau^\sigma(\tau)$.

As a special case of the above notion, we recover back the notion of a (conical) complex introduced by Kempf, Knudsen, Mumford and Saint-Donat:

Definition 4.3.2. A *complex* is a semicomplex Σ such that every $\sigma \in \Sigma$ is a fan.

Definition 4.3.3. A semicomplex Σ is called a Γ-*semicomplex* if there is a collection of algebraic groups $\Gamma_\sigma \subset T_\sigma$, where $\sigma \in \Sigma$, such that

1. Each $\sigma \in \Sigma$ is a Γ_σ-semicone.
2. For any $\tau \leq \sigma$, $\Gamma_\tau \subset \Gamma_\sigma$ and the morphism $\imath_\tau^\sigma$ induces a Γ_τ-equivariant morphism $X_\tau \to X_\sigma$. Moreover $\Gamma_\tau = (\Gamma_\sigma)_\tau = \{g \in \Gamma_\sigma \mid gx = x, x \in O_\tau\}$.

Remarks.   1. Each semicomplex can be considered as a Γ-semicomplex with trivial groups Γ_σ
2. From now on we shall often identify vectors of σ with their images under $\imath_\tau^\sigma$ if $\tau \leq \sigma$, and simplify the notation replacing $\imath_\tau^\sigma$ with set-theoretic inclusions.
3. In what follows we shall write $\sigma \preceq \sigma'$ means that σ is a face of the cone σ', while $\sigma \leq \sigma'$ to mean that σ is a face of the semicone σ'.
4. In the case when Σ is a conical complex the relations $\leq$ and $\preceq$ coincide.
5. Note that cones in a semicomplex intersect along a union of common faces.

Denote by $\mathrm{Aut}(\sigma)$ the group of automorphisms of the Γ_σ-semicone σ.

Lemma 4.3.4. *For any Γ-semicomplex Σ and any $\varrho \leq \tau \leq \sigma$ in Σ, there is an automorphism $\alpha_\varrho \in \mathrm{Aut}(\varrho)$ such that $\imath_\tau^\sigma \circ \imath_\varrho^\tau = \imath_\varrho^\sigma \alpha_\varrho$.*

Proof. By definition, the maps $\imath_\tau^\sigma \circ \imath_\varrho^\tau$ and $\imath_\varrho^\sigma$ are both linear isomorphisms of ϱ onto the image $\imath_\tau^\sigma \circ \imath_\varrho^\tau(\varrho) = \imath_\varrho^\sigma(\varrho)$. Thus $\alpha_\varrho := (\imath_\varrho^\sigma)^{-1} \imath_\tau^\sigma \circ \imath_\varrho^\tau$. $\qquad\square$

Remarks.   1. It follows from Lemma 4.3.4 that vectors in σ are in general defined up to automorphisms from $\mathrm{Aut}(\sigma)$. Hence the notion of support of a semicomplex as a topological space which is the totality of such vectors is not well defined. However if we consider, for instance, vectors invariant under all such automorphisms then the relevant topological space can be constructed (see notion of *stable support* 6.1.1) and plays the key role for semicomplexes and their subdivisions.

2. In further considerations we shall introduce the notion of *oriented semicomplex* (see Definition 4.11.1) which allows a smaller group of automorphisms $\mathrm{Aut}(\sigma)^0 \subset \mathrm{Aut}(\sigma)$. Consequently, the corresponding stable support of an oriented semicomplex is essentially larger. This allows one to perform more subdivisions and carry out certain important birational transformations.

Lemma 4.3.5. *A Γ-semifan Σ in N determines a Γ-semicomplex Σ^{semic} where each cone $\sigma \in \Sigma$ determines the semicone consisting of all faces $\tau \leq \sigma$ and $\Gamma_\tau = \{g \in \Gamma \mid gx = x, x \in O_\tau\}$.* $\qquad\square$

Definition 4.3.6. By an *isomorphism* of two Γ-semicomplexes $\Sigma \to \Sigma'$ we mean a face bijection $\Sigma \ni \sigma \mapsto \sigma' \in \Sigma'$ such that $\Gamma_\sigma = \Gamma_{\sigma'}$, along with a collection of face Γ_σ-isomorphisms $j_\sigma : \sigma \to \sigma'$, such that for any $\tau \leq \sigma$, there is a Γ_τ-automorphism β_τ of τ such that $j_\sigma \imath_\tau^\sigma = \imath_{\tau'}^{\sigma'} j_\tau \beta_\tau$.

4.4. **Inverse systems of affine algebraic groups.**

Definition 4.4.1. By an *affine proalgebraic group* we mean an affine group scheme that is the limit of an inverse system $(G_i)_{i \in \mathbf{N}}$ of affine algebraic groups and algebraic group homomorphisms $\phi_{ij} : G_i \to G_j$, for $j \geq i$.

Lemma 4.4.2. *Consider the natural morphism $\phi_i : G \to G_i$. Then $H_i := \phi_i(G)$ is an algebraic subgroup of G_i, all induced morphisms $H_j \to H_i$ for $i \leq j$ are epimorphisms and $G = \lim_\leftarrow G_i = \lim_\leftarrow H_i$. In particular $K[H_i] \subset K[H_{i+1}]$ and $K[G] = \bigcup K[H_i]$.*

Proof. The set $H_i' := \bigcap_{j>i} \phi_{ji}(G_j)$ is an intersection of algebraic subgroups of G_i. Hence it is an algebraic subgroup of G_i. Note that the induced homomorphisms $\phi_{ij}^H : H_j' \to H_i'$ are epimorphisms and that $G = \lim_\leftarrow G_i = \lim_\leftarrow H_i'$. Consequently, $\phi_i(G) = H_i = H_i'$. $\qquad\square$

Lemma 4.4.3. *The set G^K of K-rational points of G is an abstract group which is the inverse limit $G^K = \lim_\leftarrow G_i^K$ in the category of abstract groups.*

Proof. By the previous lemma we can assume all morphisms $G_i \to G_j$ to be epimorphisms. Any K-rational point x in G is mapped to K-rational points x_i in G_i. This gives an abstract group homomorphism $\phi : G^K \to \lim_\leftarrow G_i^K$. We also have $K[G] = \bigcup K[G_i]$. We have to show that any point x of $\lim_\leftarrow G_i^K$ determines a unique point y in G^K. The point x determines a sequence of maximal ideals $m_i \in K[G_i]$ such that $m_i \subset m_{i+1}$. Let $m := \bigcup m_i$. Then $m \cap K[G_i] = m_i$. Let $f \in K[G]$. Then $f \in K[G_i]$ for some i and $f - k \in m_i$ for some $k \in K$. Hence $f \in k + m$, which means that $K[G]/m = K$. The ideal m defines a K-rational point y on G^K which is mapped to x. Note that the point y is unique since there is a unique ideal m for which $m \cap K[G_i] = m_i$. $\qquad\square$

By abuse of notation we shall identify G with G^K.

4.5. **Group of automophisms of the completion of a local ring.** Let X be a stratified noetherian scheme over $\mathrm{Spec}(K)$. Let S denote the stratification and $x \in X$ be a closed K-rational point. Let $\widehat{\mathcal{O}}_{X,x}$ denote the completion of the local ring of x on the scheme X. Let $u_1, \ldots u_k \in \widehat{\mathcal{O}}_{X,x}$ generate the maximal ideal $m_{x,X} \subset \widehat{\mathcal{O}}_{X,x}$. Then $\widehat{\mathcal{O}}_{X,x} = K[[u_1, \ldots, u_k]]/I$, where I is the defining ideal. Set $\widehat{X}_x := \mathrm{Spec}(\widehat{\mathcal{O}}_{X,x})$ and $X_x^{(n)} := \mathrm{Spec}(\mathcal{O}_{X,x}/m_{x,X}^{n+1})$ for $n \in \mathbf{N}$.

Let $\mathrm{Aut}(\widehat{X}_x, S)$ (respectively $\mathrm{Aut}(X_x^{(n)}, S)$) denote the automorphism group of $\widehat{X}_x$ (resp. $X_x^{(n)}$) preserving all closures $\overline{s} \ni x$ of strata $s \in S$. (resp. preserving all subschemes $\mathrm{Spec}(\mathcal{O}_{X,x}/(I_{\overline{s}} + m_{x,X}^{n+1}))$, for all $s \in S$)

If Γ acts on X denote by $\mathrm{Aut}^\Gamma(\widehat{X}_x, S)$ (respectively $\mathrm{Aut}^\Gamma(X_x^{(n)}, S)$) the group of all Γ_x-equivariant atomorphisms of $(\widehat{X}_x, S)$ (resp. $(X_x^{(n)}, S)$).

In further considerations set $G := \mathrm{Aut}(\widehat{X}_x, S)$ (resp. $G := \mathrm{Aut}^\Gamma(\widehat{X}_x, S)$) in the case of the action of Γ, and $G_n := \mathrm{Aut}(X_x^{(n)}, S)$ (resp. $G_n := \mathrm{Aut}^\Gamma(X_x^{(n)}, S)$).

Lemma 4.5.1. 1. G_n is an algebraic group acting on $X_x^{(n)}$. That is there exists a co-action

$$\Phi_n^* : \mathcal{O}_{X,x}/m_{x,X}^{n+1} \to K[G_n] \otimes \mathcal{O}_{X,x}/m_{x,X}^{n+1}$$

defining the action morphism $\Phi_n : G_n \times X^{(n)} \to X^{(n)}$
and an action automorphism $\Psi_n := \pi_n \times \Phi_n : G_n \times X^{(n)} \to G_n \times X^{(n)}$
where π_n is the projection on G_n.
2. The G_n form an inverse system of algebraic groups and $G = \lim_\leftarrow G_n$ is a proalgebraic group.
3. The co-actions in (1) define a ring homomorphism $\Phi^* : \widehat{\mathcal{O}}_{X,x} \to K[G][[\widehat{\mathcal{O}}_{x,X}]]$
and a _formal action morphism_ $\Phi : G\widehat{\times}\widehat{X}_x := \mathrm{Spec}\, K[G][[\widehat{\mathcal{O}}_{X,x}]] \to \widehat{X}_x$
such that for any $g \in G^K$ the restriction $\Phi_{|\{g\}\widehat{\times}\widehat{X}_x} : \{g\}\widehat{\times}\widehat{X}_x \to \widehat{X}_x$ is given by the action of g.
4. The automorphisms Ψ_n of $X^{(n)}$ define a _formal action automorphism_ Ψ of
$$G\widehat{\times}\widehat{X}_x = \lim_\leftarrow G \times X^{(n)}$$
such that for any $g \in G^K$ the restriction $\Psi_{|\{g\}\widehat{\times}\widehat{X}_x} : \{g\}\widehat{\times}\widehat{X}_x \to \{g\}\widehat{\times}\widehat{X}_x$ is the automorphism given by the action of g.
5. Let $u_1, \ldots, u_k$ denote the local parameters on $\widehat{X}_x$. There exist regular functions $g_{\alpha\beta}$ generating $K[G]$ such that the action is given by $u^\alpha \mapsto \sum g_{\alpha\beta}u^\beta$, where $\alpha, \beta \in \mathbb{Z}_{\geq 0}^n$ are multiindices.

Proof. (1) Write $K[X_x^{(n)}] = K[u_1, \ldots, u_k]/(I + m_x^{n+1})$. Set $W^n := K[u_1, \ldots, u_k]/(m_x^{n+1})$, $I^n := (I + m_x^{n+1})/m_x^{n+1}$, $I_s^n := (I_s + m_x^{n+1})/m_x^{n+1}$.
Define the product $W^n \times W^n \to W^n$ to be the bilinear map for which $u^{\alpha_1} \cdot u^{\alpha_2} = u^{\alpha_1+\alpha_2}$. Set

$$H_n := \{g \in \mathrm{Gl}(W^n) \mid g(u^{\alpha_1} \cdot u^{\alpha_2}) = g(u^{\alpha_1}) \cdot g(u^{\alpha_2})\}.$$

H_n is an algebraic subgroup of $\mathrm{Gl}(W^n)$. Then

$$G_n = \{g \in H^n \mid g(I^n) = I^n, g(I_s^n) = I_s^n, \quad \text{for all} \quad s \in S, ga = ag \quad \text{for all} \quad a \in \Gamma_x\}$$

is algebraic. The embedding $G_n \subset \mathrm{Gl}(W^n)$ determines the natural co-action $K[W^n] \to K[G_n] \otimes K[W_n]$,

$$u^\alpha \mapsto \sum_{\beta \in \mathbf{Z}_{\geq 0}^n} g_{\alpha\beta}u^\beta,$$

which factors to $K[X_x^{(n)}] \to K[G_n] \otimes K[X_x^{(n)}]$.

(2) If the morphism $X_x^{(n)} \hookrightarrow X_x^{(n+1)}$ commutes with automorphisms ,i.e. if Φ is an automorphism of $X_x^{(n+1)}$, then $\Phi_{|X^{(n)}} : X^{(n)} \to X^{(n)} \subset X_x^{(n+1)}$ is an automorphism of $X^{(n)}$. This defines the morphisms $\mathrm{Aut}(X_x^{n+1}, S) \to \mathrm{Aut}(X_x^{(n)}, S)$.

The K-rational points of the proalgebraic group $G = \lim_\leftarrow G_n$ can be identified, by Lemma 4.4.3, with (Γ_x-equivariant) automorphisms of $\widehat{X}_x$ preserving strata.

(3) The morphisms $\widehat{\mathcal{O}}_{X,x} \to K[G_n][[\mathcal{O}_{x,X}]]/m_{x,X}^n \to K[G][[\widehat{\mathcal{O}}_{X,x}]]/m_{x,X}^n$ determine
$$\widehat{\mathcal{O}}_{X,x} \to \lim_\leftarrow K[G][[\widehat{\mathcal{O}}_{x,x}]]/m_{x,X}^n$$

(4) and (5) follow from (1) $\qquad\square$

Lemma 4.5.2. Let $G = \lim_\leftarrow G_i$ be a connected proalgebraic group acting on $\widehat{X}_x$. Let $\Phi : G\widehat{\times}\widehat{X}_x \to \widehat{X}_x$ be the action morphism and $\pi : G\widehat{\times}\widehat{X}_x \to \widehat{X}_x$ be the standard projection. Then for any ideal I on $\widehat{X}_x$ the following conditions are equivalent:

- I is invariant with respect to the action of the abstract group G^K
- $\Phi^*(I) = \pi^*(I)$

Proof. It suffices to prove the equivalence of the conditions for any scheme $X_x^{(n)}$ and ideal $I_n = I \cdot K[X_x^{(n)}]$. In this situation the assertion reduces to the well known case of an algebraic action of G_n on $X_x^{(n)}$. If

$\Phi^*(I_n) = \pi^*(I)$ then for any $g \in G_n^K$ and $f \in I_n$, we have $g \cdot f = \Phi_g^*(f) \in \pi_g^*(I_n) = I_n$. Thus I_n is G_n^K-invariant.

Now suppose I_n is G_n^K-invariant. Let $\Phi^*(f) = \sum f_i g_i$ for $f \in I_n$. We can assume that $g_i \in K[G_n]$ are linearly independent. Hence we can find elements $g^i \in G_n^K$ such that $v_i := (g_i(g^1), \ldots, g_i(g^l))$ are linearly independent vectors for $i = 1, \ldots, l$. Then we find coefficients α_{ij} such that $\sum_j \alpha_{ij} v_j = (0, \ldots, 1_j, \ldots, 0)$. Then $f_i = \sum_j \alpha_{ij} f_i g_i(g^j) = \sum_j \alpha_{ij}(g^j \cdot f_i) \in I_n$. Thus for any $f \in I_n$, $\Phi^*(f) \in \pi^*(I_n)$, which gives $\Phi^*(I) \subset \pi^*(I)$.

Denote by inv the automorphism of $G \widehat{\times} \widehat{X}_x$ induced by taking the inverse $g \mapsto g^{-1}$. Then $\pi^*(I) = \Psi^{-1*}\Phi^*(I) \subset \Psi^{-1*}\pi^*(I) = \Psi^*\text{inv}^*\pi^*(I) = \Psi^*\pi^*(I) = \Phi^*(I)$.   □

Let L denote an algebraically closed field containing K. Set

$$X_x^{(n)L} := X_x^{(n)} \times_{\text{Spec}\,K} \text{Spec}\,L,$$

$$\widehat{X}_x^L := \varprojlim X_x^{(n)L} = \text{Spec}\,L[[\mathcal{O}_{X,x}]]$$

Let $\text{Aut}_L^\Gamma(X_x^{(n)L}, S)$ denote the group of Γ_x-equivariant L-automorphisms of $X_x^{(n)L}$ preserving the relevant ideals of strata and $G := \text{Aut}_L(\widehat{X}_x^L, S)$ denote the group of L-automorphisms of $\widehat{X}_x^L$.

Lemma 4.5.3.   1. $\text{Aut}_L^\Gamma(X_x^{(n)L}, S) = \text{Aut}_K^\Gamma(X_x^{(n)}, S)) \times_{\text{Spec}\,K} \text{Spec}\,L$.
  2. $\text{Aut}_L^\Gamma(\widehat{X}_x^L, S) = \text{Aut}_K^\Gamma(\widehat{X}_x, S) \times_{\text{Spec}\,K} \text{Spec}\,L$.
  3. *Let* $G \subset \text{Aut}_K(\widehat{X}_x, S)$ *be a proalgebraic subgroup. An ideal* $I \subset \widehat{\mathcal{O}}_{X,x}$ *is* G-*invariant iff* $I \cdot L[[\mathcal{O}_{X,x}]]$ *is* $G \times_{\text{Spec}\,K} \text{Spec}\,L$-*invariant.*   □

Proof. (1) and (2) follow from the construction of $\text{Aut}_K^\Gamma(\widehat{X}_x, S)$.

(3)($\Rightarrow$) It follows from Lemma 4.5.2 that $I \subset \widehat{\mathcal{O}}_{X,x}$ is G-invariant iff $\Phi^*(I) = \pi^*(I)$ or $\Psi^*(\frac{I+m_x^n}{m_x^n}) = \pi^*(\frac{I+m_x^n}{m_x^n})$. Let Ψ^L and π be pull-backs of the morphisms Ψ and π^L under $\text{Spec}\,L \to \text{Spec}\,K$. Then $\Psi^{L*}(\frac{I+m_x^n}{m_x^n} \cdot L) = \pi^{L*}(\frac{I+m_x^n}{m_x^n} \cdot L) \subset L \otimes_K \widehat{\mathcal{O}}_{X,x}/m_x^n$, which shows , by Lemma 4.5.2, that $I \cdot L[[\mathcal{O}_{X,x}]]$ is $G \times_{\text{Spec}\,K} \text{Spec}\,L$-invariant.

($\Leftarrow$). Assume the latter holds then $\Psi^{L*}(\frac{I+m_x^n}{m_x^n} \cdot L) = \pi^{L*}(\frac{I+m_x^n}{m_x^n} \cdot L) \subset L \otimes_K \widehat{\mathcal{O}}_{X,x}/m_x^n$. Let $\text{Gal}(L/K)$ be the Galois group of the extension $K \subset L$. Both homomorphisms π^{L*} and Ψ^{L*} are $\text{Gal}(L/K)$-equivariant. Considering $\text{Gal}(L/K)$-invariant elements gives $\Psi^*(\frac{I+m_x^n}{m_x^n}) = \pi^*(\frac{I+m_x^n}{m_x^n})$.   □

Example 4.5.4. Let $\phi_n : \text{Aut}(\widehat{X}_x, S) \to \text{Aut}(X_x^{(n)}, S)$ denote the natural morphisms. For $n = 1$ we get the differential mapping:

$$d = \phi_1 : \text{Aut}(\widehat{X}_x, S) \longrightarrow \text{Aut}(X_x^{(1)}, S) \subset \text{Gl}(\text{Tan}_{X,x}).$$

4.6. **Proof of Demushkin's Theorem.** In what follows we shall use the following generalization of Theorem 2.5.1.

Lemma 4.6.1. *Let* σ *and* τ *be two* Γ-*semicones in isomorphic lattices* $N_\sigma \simeq N_\tau$.

Then there exists a Γ-*equivariant isomorphism* $\widehat{X}_\sigma \simeq \widehat{X}_\tau$ *preserving strata iff there exists an isomorphism of* Γ-*semicones* $\sigma \simeq \tau$.

For a Γ-semicone σ in a lattice N_σ denote by $\text{Aut}(\widehat{X}_\sigma)$ the group of all Γ-equivariant automorphisms of $\widehat{X}_\sigma$ which preserve strata defined by faces of the Γ-semicone σ.

Lemma 4.6.2. *(Demushkin [22], also Gubeladze [27]) Let* σ *and* τ *be two* Γ-*semicones in isomorphic lattices* $N_\sigma \simeq N_\tau$.

  1. *The torus* T_σ *associated to* $\widehat{X}_\sigma$ *determines a maximal torus in the proalgebraic group* $G := \text{Aut}(\widehat{X}_\sigma)$.
  2. *Let* $d : G \to \text{Gl}(\text{Tan}_{X_\sigma, O_\sigma})$ *be the differential morphism as in Example 4.5.4. Then* $d(T_\sigma)$ *is a maximal torus in* $d(G)$.
  3. *Any* Γ-*equivariant isomorphism* $\widehat{\phi} : \widehat{X}_\sigma \simeq \widehat{X}_\tau$, *preserving strata determines an action of the torus* T_τ *on* $\widehat{X}_\sigma$ *and a group embedding* $T_\tau \hookrightarrow G$. *If* $d(T_\sigma) = d(T_\tau)$ *then* T_σ *and* T_τ *are conjugate in* G.
  4. *Any two tori* $T_\sigma, T_\tau \subset G$ *which are determined by* $\widehat{\phi}$, *are conjugate in* G.

Proof. (1) The torus T_σ is maximal in G since its centralizer consists of T_σ-equivariant automorphisms of $\widehat{X}_\sigma$, preserving strata and therefore it coincides with T_σ (T_σ-equivariant automorphisms multiply characters by constants and are defined by elements of T_σ); (see also Step 4 in [22]).

(2) By Lemma 4.4.2 we can write $G = \lim_\leftarrow G_i$, where all homomorphisms $G_i \to G_j$ are epimorphisms. We show that the relevant images of T_σ are maximal tori in all groups G_i. Otherwise we find maximal tori $T_i \subset G_i$ containing the images of T_σ, such that $\lim_\leftarrow T_i$ is a maximal torus in G containing T_σ as a proper subgroup (see [10] Proposition 11.14(1)); (see also step 9 in [22]).

(3) Let $T_\sigma \subset G$ be the maximal torus associated to $\widehat{X}_\sigma$. Let $u_1, \dots, u_k$ be characters of T_σ generating the maximal ideal of O_σ and $x_1, \dots, x_k$ be semiinvariant functions determined by the action of T_σ, induced on the tangent space $\mathrm{Tan}_{O_\sigma, X_\sigma}$. Characters of T_σ define the natural T_σ-equivariant embedding $\phi_\sigma : \widehat{X}_\sigma \to \mathrm{Tan}_{O_\sigma, X_\sigma}$ into the tangent space induced by the ring epimorphism $\phi_\sigma^* : K[[x_1, \dots, x_k]] \to K[[u_1, \dots, u_k]]$ sending x_i to u_i. The morphism ϕ_σ determines the isomorphism $\widehat{\phi}_\sigma : \widehat{X}_\sigma \to \widehat{\phi}_\sigma(\widehat{X}_\sigma)$ preserving strata corresponding to the isomorphism $\widehat{\phi}_\sigma^* : K[[x_1, \dots, x_k]]/I \to K[[u_1, \dots, u_k]]$ where the ideal I is generated by all elements $x^\alpha - x^\beta$ where x^α, x^β are monomials on which T_σ acts with the same weights.

If $d(T_\sigma) = d(T_\tau)$ then $\phi_\sigma(\widehat{X}_\sigma) = \phi_\tau(\widehat{X}_\tau) = \mathrm{Spec}\, K[[x_1, \dots, x_k]]/I$. Consequently, $\beta := \phi_{T_\sigma}^{-1} \phi_{T_\sigma}$ is a $T_\sigma = d(T_\sigma)$-equivariant isomorphism $\widehat{X}_\sigma \to \widehat{X}_\tau$ preserving strata. It follows that β determines a conjugating automorphism with $T_\sigma = \beta T_\tau \beta^{-1}$.

(4) By Borel's theorem ([10]) and (2), the tori $d(T_\sigma)$ and $d(T_\tau)$ are conjugate in $d(G)$. Consequently, we can find a torus T conjugate to T_σ and such that $d(T_\tau) = d(T)$. By (3) T_τ and T are conjugate. $\square$

Proof of Lemma 4.6.1. Let $\widehat{\phi} : \widehat{X}_\sigma \to \widehat{X}_\tau$ be a Γ-equivariant isomorphism preserving strata. By Lemma 4.6.2 there is a Γ-equivariant automorphism α of $\widehat{X}_\sigma$ for which $T_\sigma = \alpha T_\tau \alpha^{-1}$. Then T_σ acts on $\widehat{X}_\tau$ as a big torus and the Γ-equivariant isomorphism $\widehat{\phi} \circ \alpha^{-1} : \widehat{X}_\sigma \to \widehat{X}_\tau$ is T_σ-equivariant and preserves strata. Such an isomorphism yields an isomorphism of the associated Γ-semicones. $\square$

4.7. Singularity type of nonclosed points.

For a ring L and a cone σ in N_σ denote by $L[[\sigma^\vee]]$ the completion of the ring $L[\sigma^\vee]$ at the ideal m_σ of O_σ.

Lemma 4.7.1. *Let L be a ring containing K, and σ and τ be Γ-semicones in isomorphic lattices $N_\sigma \simeq N_\tau$. Let m_σ, m_τ denote the ideals of O_σ and O_τ respectively. The action of Γ on $K[[\sigma^\vee]]$ and $K[[\tau^\vee]]$ induces an action on $L[[\sigma^\vee]]$ and $L[[\tau^\vee]]$ which is trivial on L.*

Any Γ-equivariant isomorphism $\phi : L[[\sigma^\vee]] \simeq L[[\tau^\vee]]$ for which $\phi(m_\sigma) = m_\tau$ can be decomposed as $\phi = \phi_0 \phi_1$, where ϕ_0 is a Γ-equivariant automorphism of $L[[\tau^\vee]]$ identical on monomials and ϕ_1 is a Γ-equivariant L-isomorphism.

Proof. (1) The restriction of ϕ to L is a ring monomorphism $\phi_{|L} : L \to L[[\tau^\vee]]$. This ring homomorphism can be extended to a ring homomorphism $\phi_0 : L[[\tau^\vee]] \to L[[\tau^\vee]]$ by sending monomials identically to the same monomials, i.e. $\phi_0(\sum_{\alpha \in \tau^\vee} a_\alpha x^\alpha) = \sum_{\alpha \in \tau^\vee} \phi_0(a_\alpha) x^\alpha$, where $\phi_0(a_\alpha) = \sum_{\beta \in \tau^\vee} a_{\alpha\beta} x^\beta$. Note that in the expression $\sum_{\alpha \in \tau^\vee} \phi_0(a_\alpha) x^\alpha$, the coefficient of x^α is a finite sum since there are finitely many possibilities to express α as $\alpha' + \beta$ where $\alpha', \beta \in \tau^\vee$. Thus the above expression defines a formal power series and consequently ϕ_0 is a ring homomorphism. Then $\phi_{0|L}$ determines an automorphism $\phi_1 : L \to L[[\tau^\vee]]/m_\tau = L$. Let ϕ_2 denote the automomorphism of $L[[\tau^\vee]]$ induced by the ϕ_1 and identical on monomials. The composition $\psi = \phi_2^{-1} \phi_0$ is a ring endomorphism which as a K-linear transformation can be written in the form $\psi = id + p$, where $p(m^i) \subset m^{i+1}$. Such a linear transformation is invertible (with inverse $id - p + p^2 - \dots$), hence ψ is a linear isomorphism and ring isomorphism. This implies that ϕ_0 is an automorphism and $\phi_1 := \phi_0^{-1} \phi$ is an L-isomorphism.

Note that the above morphisms are Γ-equivariant. $\square$

The following is a generalization of the Demushkin Lemma.

Lemma 4.7.2. *Let σ and τ be Γ-semicones in isomorphic lattices $N_\sigma \simeq N_\tau$, and L be a field containing K. Consider the induced action of Γ on $L[[\sigma^\vee]]$ and $L[[\tau^\vee]]$ which is trivial on L. There exists a Γ-equivariant isomorphism $L[[\sigma^\vee]] \simeq L[[\tau^\vee]]$ over K, preserving strata iff σ and τ are Γ-isomorphic.*

Proof. Denote by ϕ an isomorphism $L[[\sigma^\vee]] \overset{\phi}{\simeq} L[[\tau^\vee]]$. By Lemma 4.7.1, $\phi = \phi_0\phi_1$, where ϕ_1 is an L-isomorphism. Tensoring with the algebraic closure $\overline{L}$ of L and taking completion determines an $\overline{L}$-isomorphism $\overline{\phi}_1 : \overline{L}[[\sigma^\vee]] \simeq \overline{L}[[\tau^\vee]]$. It suffices to apply Theorem 2.5.4 to the above $\overline{L}$-isomorphism

Note that the above isomorphisms are Γ-equivariant. $\square$

This lemma allows us to extend the notion of singularity type to any nonclosed points on toric varieties X_Δ and even on some other toroidal schemes.

Lemma 4.7.3. *Let Δ be a subdivision of σ and $\widehat{X}_\Delta := X_\Delta \times_{X_\sigma} \widehat{X}_\sigma$ be a toroidal scheme with an action of a group $\Gamma \subset T_\sigma$. Let $p \in Y = \widehat{X}_\Delta$ be a point which need not be closed. Set $\Gamma_p := \{g \in \Gamma \mid g(p) = p, g_{|K_p} = \mathrm{id}_{|K_p}\}$.*

1. *Let $O_{\tau,Y} \in Y$ denote the locally closed subscheme defined by a toric orbit $O_\tau \subset X_\Delta$, where $\tau \in \Delta$. Then $\widehat{\mathcal{O}}_{O_\tau,Y} \simeq K_{O_\tau}[[\underline{\tau}^\vee]]$, where K_{O_τ} is the residue field of O_τ.*
2. *There is a Γ_p-equivariant isomorphism $\widehat{\mathcal{O}}_{p,Y} \simeq K_p[[\sigma_p^\vee]]$, where K_p is the residue field of p and σ_p is a uniquely determined cone. Moreover $\mathrm{sing}^\Gamma(p) := (\Gamma_p, \underline{\mathrm{sing}^{\Gamma_p}(\sigma_p)}) = \mathrm{sing}^\Gamma(\tau) := (\Gamma_\tau, \underline{\mathrm{sing}^{\Gamma_\tau}(\tau)})$, where $O_{\tau,Y} \in Y$ is the minimal orbit scheme containing p.*

Proof. (1) Let $X_{\tau,Y} := X_\tau \times_{X_\sigma} \widehat{X}_\sigma \subset Y$. Then $K[X_{\tau,Y}] = K[\tau^\vee] \otimes_{K[\sigma^\vee]} K[[\sigma^\vee]]$. Let $T_\tau \subset T$ be the torus corresponding to the sublattice $N_\tau := N \cap \mathrm{lin}(\sigma)$. Then T_τ acts on $K[X_{\tau,Y}]$ with nonnegative weights. Therefore the subring of T_τ-invariant functions $K[X_{\tau,Y}]^{T_\tau}$ in $K[X_{\tau,Y}]$ equals $K[\tau^\perp] \otimes_{K[\sigma^\vee \cap \tau^\perp]} K[[\sigma^\vee \cap \tau^\perp]]$. The ideal $I = I_{O_{\tau,Y}} \subset K[X_{\tau,Y}]$ of the orbit $O_{\tau,Y}$ is generated by all monomials with positive weights in the set $\sigma^\vee + \tau^\vee \subset \tau^\vee = \underline{\tau}^\vee \oplus \tau^\perp$ and consequently it is generated by $\underline{\tau}^\vee \setminus \{0\}$. Thus $K[X_{\tau,Y}]^{T_\tau} \simeq K[O_{\tau,Y}]$ and $\lim_{\leftarrow} K[X_{\tau,Y}]/I^k \simeq K[O_{\tau,Y}][[\underline{\tau}^\vee]]$. This gives $K[\widehat{Y}_{O_{\tau,Y}}] \simeq K(O_{\tau,Y})[[\underline{\tau}^\vee]]$.

(2) Let $O_{\tau,Y} \in Y$ be the minimal orbit scheme containing p. First we prove that $\Gamma_{O_\tau} = \Gamma_p$. We have the obvious inclusion $\Gamma_{O_\tau} \subset \Gamma_p$. Now if $g \in \Gamma_p$ then for any $f \in K[O_\tau]$, $(g(f) - f) \in I_p$, where $I_p \subset K[O_\tau]$ describes p. Since $\Gamma_p = \Gamma_{p'}$ for any $p' \in \Gamma \cdot p$, it follows that $(g(f) - f) \in \bigcap_{h \in T} h \cdot I_p = \{0\}$.

The ideal $I_p \subset \mathcal{O}_{p,O_\tau}$ of p is generated by local parameters $u_1, \ldots, u_l$. Then there are Γ_p-equivariant isomorphisms $\widehat{\mathcal{O}}_{p,O_\tau} \simeq K_p[[u_1, \ldots, u_l]]$ and $\widehat{\mathcal{O}}_{p,Y} \simeq K_p[[u_1, \ldots, u_l]][[\tau^\vee]] = K_p[[(\tau \oplus^{\Gamma_p} \langle e_1, \ldots, e_l \rangle)^\vee]]$ where $\widehat{\mathcal{O}}_{p,O_\tau} = \widehat{\mathcal{O}}_{p,Y}^{T_\tau}$. Therefore $\sigma_p = \tau \oplus^{\Gamma_p} \langle e_1, \ldots, e_l \rangle$ and $\underline{\mathrm{sing}^{\Gamma_p}(\sigma_p)} = \underline{\mathrm{sing}^{\Gamma_\tau}(\tau)}$. $\square$

By Lemmas 4.7.2 and by 4.7.3 the singularity type of a nonclosed point p, $\mathrm{sing}(p) := \mathrm{sing}(\sigma_p)$ (resp. $\mathrm{sing}^{\Gamma_p}(p) := (\Gamma_p, \underline{\mathrm{sing}^{\Gamma_p}(\sigma_p)})$, is uniquely determined. Moreover singularity type determines a stratification $\mathrm{Sing}(Y)$ (resp. $\mathrm{Sing}^\Gamma(Y)$) on Y such that all points in the same stratum have the same singularity type. This yields

Lemma 4.7.4. *Let Γ act on $\widehat{X}_\Delta := X_\Delta \times_{X_\sigma} \widehat{X}_\sigma$ There is a stratification $\mathrm{Sing}^\Gamma(\widehat{X}_\Delta)$ of $\widehat{X}_\Delta$ which is determined by singularity type and therefore preserved by any Γ-equivariant automorphism of $\widehat{X}_\Delta$.* $\square$

4.8. Semicomplex associated to a stratified toroidal variety. For a Γ-semicone σ we denote by X_σ the associated stratified toric variety.

Definition 4.8.1. Let (X, S) be a Γ-stratified toroidal variety. We say that a semicomplex Σ is *associated* to (X, S) if there is a bijection $\Sigma \to S$ with the following properties: Let $\sigma \in \Sigma$ map to $s = \mathrm{strat}_X(\sigma) \in S$. Then any $x \in s$ admits an open Γ-invariant neighborhood $U_\sigma \subset X$ and a Γ-equivariant Γ_s-smooth morphism $\phi_\sigma : U_\sigma \to X_\sigma$ of stratified varieties such that $s \cap U$ equals $\phi_\sigma^{-1}(O_\sigma)$ and the intersections $s' \cap U$, $s' \in S$, are precisely the inverse images of the strata of X_σ and the action of Γ on X_σ extends the action of Γ_s.

We call the smooth morphisms $U_\sigma \to X_\sigma$ from the above definition *charts*. A collection of charts satisfying the conditions from the above definition is called an *atlas*.

Remarks.   1. Different (X, S) may have the same associated semicomplex Σ. Smooth varieties endowed with the trivial stratification have the associated semicomplex consisting of one zero cone.
      2. The action of $\Gamma_s = \Gamma_\sigma$ is fixed for σ while the action of Γ on X_σ depends upon charts.

Lemma 4.8.2. *For any Γ-stratified toroidal variety (X, S) there exists a unique (up to isomorphism) associated Γ-semicomplex Σ. Moreover $\tau \leq \sigma$ iff $\overline{\mathrm{strat}_X(\tau)} \supset \mathrm{strat}_X(\sigma)$. (X, S) is a toroidal embedding iff Σ is a complex.*

Proof First we assign to any stratum s a semicone σ.

By Definition 4.1.5 there is a Γ_s-smooth morphism $\phi_\sigma : U_\sigma \to X_\sigma$ into a stratified toric variety (X_σ, S_σ) preserving strata. Note that Γ_s acts trivially on O_σ. So we can assume that $\sigma = \underline{\sigma}$ is of maximal dimension in N_σ by composing ϕ, if necessary, with a suitable projection.

We define the Γ_s-semicone σ to be the Γ_s-semifan associated to the Γ_s-stratified toric variety (X_σ, S_σ).

Then there is an open subset U of U_σ intersecting s and a Γ_s-étale morphism $\phi := \phi_\sigma \times \psi : U \to X_\sigma \times \mathbf{A}^k = X_\tau$ preserving strata, where $\tau = \sigma \times \langle e_1, \ldots, e_{\dim(s)} \rangle$, and $\psi : U \to \mathbf{A}^k$ is a morphism defined by local parameters $u_1, \ldots, u_k$ on s. The stratification of X_τ is defined by the embedded semifan $\sigma \subset \overline{\tau}$, consisting of the faces of the semicone σ in the lattice N_τ. Then $\widehat{\phi} : \widehat{X}_x \to \widehat{X}_\tau$ is a Γ_s-equivariant isomorphism preserving strata. By Lemma 4.6.1 the embedded Γ_s-semifan $\sigma \subset \tau$ and the Γ_s-semicone σ are determined uniquely up to isomorphism. We write $s = \mathrm{strat}_X(\sigma)$ and define $\Gamma_\sigma := \Gamma_s$.

Assume that the closure of a stratum $s = \mathrm{strat}_X(\sigma)$ contains a stratum $t = \mathrm{strat}_X(\tau)$. Let $\phi_\tau : U_\tau \to X_\tau$ denote a chart associated with τ. Then the stratum $s \cap U_\tau$ and the strata of U_τ having $s \cap U_\tau$ in their closure determine a semicone $\sigma_s \subset \overline{\tau}$. It follows from the uniqueness of σ that there is a saturated embedding i_σ^τ of the semicone σ into the semicone τ with the image $i_\sigma^\tau(\sigma) = \sigma_s$. Then we shall write $\sigma \leq \tau$.

The Γ-semicomplex Σ is defined as the collection of the Γ_σ-semicones σ, and the saturated face inclusions i_σ^τ for $\sigma \leq \tau$.

Now let Σ and Σ' be two semicomplexes associated to (X, S). By uniqueness there are isomorphisms of semicones $j_\sigma : \sigma \to \sigma'$ (see Definition 4.3.6). These isomorphisms induce an isomorphism of semicomplexes.

The second part follows from the fact that locally toroidal embeddings correspond to fans consisting of all faces of some cone. $\qquad\square$

Lemma 4.8.3. *Let (X_Σ, S) be a Γ-stratified toric variety corresponding to an embedded Γ-semifan $\Omega \subset \Sigma$. Then (X_Σ, S) is a Γ-stratified toroidal variety with the associated Γ-semicomplex Ω^{semic}.*

There is an atlas

$$\mathcal{U}^{\mathrm{can}}(X_\Sigma, S) = \mathcal{U}(\Sigma, \Omega)$$

on (X_Σ, S) defined as follows: For any σ in N such that $\omega(\sigma) = \omega$ in N_ω there is a chart $\phi_\sigma : X_\sigma \to X_\omega$ given by any projection $\pi_\sigma^\omega : \sigma \to \omega$ such that $\pi_{\sigma|\omega}^\omega = \mathrm{id}_{|\omega}$.

Proof. Follows from Proposition 3.1.7 and from the definition of the associated semicomplex. $\qquad\square$

4.9. Local properties of orientation.

Definition 4.9.1. We shall call a proalgebraic group G *connected* if it is a connected affine scheme. For any proalgebraic group $G = \lim_\leftarrow G_i$, denote by G^0 its maximal connected proalgebraic subgroup $G^0 = \lim_\leftarrow G_i^0$.

Lemma 4.9.2. *$G = \lim_\leftarrow G_i$ is connected if each G_i is irreducible.*

Proof. By Lemma 4.4.2 we can assume all morphisms $G_i \to G_j$ to be epimorphisms and $K[G] = \bigcup K[G_i]$. If $g_1, g_2 \in K[G]$ and $g_1 \cdot g_2 = 0$ then $g_1, g_2 \in K[G_i]$ for some i and g_1 or g_2 equals zero. $\qquad\square$

Lemma 4.9.3. *Let $G = \lim_\leftarrow G_i$ be a connected proalgebraic group. Then the image $\phi_i(G) \subset G_i$ of the natural homomorphism $\phi_i : G \to G_i$ is connected.* $\qquad\square$

Definition 4.9.4. Let X be a stratified (resp. Γ-stratified) toroidal scheme over K. We say that an automorphism (resp. Γ_x-equivariant automorphism) ϕ of X *preserves orientation* at a K-rational point $x = \phi(x)$ if it induces an automorphism $\widehat{\phi} \in \mathrm{Aut}(\widehat{X}_x, S)^0$ (resp. $\widehat{\phi} \in \mathrm{Aut}^\Gamma(\widehat{X}_x, S)^0$).

Definition 4.9.5. We say that two étale morphisms (resp. Γ-étale morphisms) of stratified toroidal schemes over K, $f_1, f_2 : (X, S) \to (Y, T)$, *determine the same orientation* at a closed K-rational point $x \in X$ if $f_1(x) = f_2(x)$ and $(\widehat{f}_2)^{-1} \circ \widehat{f}_1 \in \mathrm{Aut}(\widehat{X}_x, S)^0$ (resp. $(\widehat{f}_2)^{-1} \circ \widehat{f}_1 \in \mathrm{Aut}^\Gamma(\widehat{X}_x, S)^0$).

In further considerations we shall consider the case of Γ-stratified toroidal schemes. The case of stratified toroidal schemes corresponds to the situation when Γ is a trivial group.

Definition 4.9.6. Let (X, S) is a Γ-stratified toroidal scheme and $x \in X$ be a K-rational point. We call functions $y_1, \ldots, y_k$ *locally toric parameters* if $y_1 = \phi_x^*(u_1), \ldots, y_k = \phi_x^*(u_k)$, where $u_1, \ldots, u_k$ are semiinnvariant generators at the orbit point O_σ on a Γ-stratified toric variety X_σ, and $\psi_x : U_x \to X_\sigma$ is a Γ_x-étale morphism from an open Γ_x-invariant neighborhood U_x of x such that $\psi_x(x) = O_\sigma$ and the intersections $s' \cap U$, $s' \in S$, are precisely the inverse images of strata of X_σ.

Lemma 4.9.7. *Locally toric parameters exist for any K-rational point of a Γ-stratified toroidal scheme X.*

Proof Let s be the stratum through x. By Definition 4.1.6 there is a Γ_x-śmooth morphism from an open neighborhood U_x of x such that $s = \phi_x^{-1}(O_\sigma)$ and the intersections $s' \cap U$, $s' \in S$, are precisely the inverse images of strata of X_σ. Let $x_1, \ldots, x_k$ be local paramters of s at x. Set $g : U \to \mathbf{A}^1$, where $g(x) = (x_1, \ldots, x_k)$. Then the morphism $\psi_x := \phi_x \times g$ is Γ_x-étale and defines locally toric parameters at x. $\qquad\square$

Lemma 4.9.8. *Let (X, S) be a Γ-stratified toroidal scheme and $x \in X$ be a K-rational point. Let $y_1, \ldots, y_n$ be locally toric paramters at $x \in X$. The ideals of closures of the strata $I_s \subset \mathcal{O}_{x,X}$ are generated by subsets of $\{u_1, \ldots, u_k\}$.*

Proof. It suffices to show the lemma for characters generating $\sigma^\vee$ on a Γ-stratified toric variety X_σ. Let T be the big torus on X_σ. Each stratum s is T-invariant irreducible hence contains a $\mathbf{T}$- orbit O_τ, where $\tau \preceq \sigma$. We conclude that $\overline{s} = \overline{O_\tau}$. But then $I_{\overline{O_\tau}} \subset K[X_\sigma]$ is generated by functions corresponding to those generating functionals of $\sigma^\vee$ which are not zero on τ. $\qquad\square$

Lemma 4.9.9.   1. *The group $\mathrm{Aut}^\Gamma(\widehat{X}_x, S)$ is connected for any smooth Γ-stratified toroidal scheme (X, S) over $\mathrm{Spec}(K)$ and any K-rational point $x \in X^\Gamma$.*
    2. *Any two Γ-étale morphisms $g_1 : (X, S) \to (Y, R)$, $g_2 : (X, S) \to (Y, R)$ between smooth Γ-stratified toroidal schemes such that $g_1(x) = g_2(x)$ determine the same orientation at x.*

Proof. (1) We can replace X by $\mathbf{A}^k$ since $(\widehat{X}_x, S) \simeq (\widehat{\mathbf{A}^k}_0, S_A)$ for a toric stratification S_A on $\mathbf{A}^k$. Let m be the maximal ideal of $0 \in \mathbf{A}^k$. The automorphism $g \in G := \mathrm{Aut}^\Gamma(\widehat{\mathbf{A}^k}_0, S_A)$ is defined by Γ-semiinvariant functions $g^*(x_1), \ldots, g^*(x_k)$, where $x_1, \ldots, x_k$ are the standard coordinates on $\mathbf{A}^k$ such that the Γ-weights of $g^*(x_i)$ and x_i, where $i = 1, \ldots, k$, are equal. There is a birational map $\alpha : \mathbf{A}^1 \to G$ defined by

$$\alpha(z) := (x_1, \ldots, x_k) \mapsto ((1-z)g^*(x_1) + zx_1, \ldots, (1-z)g^*(x_k) + zx_k).$$

Note that $\alpha(z)$ defines automorphisms for the open subset U of $\mathbf{A}^1$, where the linear parts of coordinates of $\alpha(z)$ are linearly independent. By Lemma 4.9.8 the closure of each stratum s is described by a subset of $\{x_1, \ldots, x_k\}$. Since g preserves strata, $\overline{s}$ is described by the corresponding subset of $\{g^*(x_1), \ldots, g^*(x_k)\}$. Thus the corresponding coordinates of $\alpha(z)$, $z \in U$, belong to the ideal $I_{\overline{s}}$. Since they are linearly independent of order 1 they generate the ideal $\frac{I_{\overline{s}}}{I_{\overline{s}} \cdot m}$ and by the Lemma of Nakayama they generate the ideal $I_{\overline{s}}$. Therefore the automorphisms $\alpha(z)$, $z \in U$, preserve strata. The morphism α "connects" the identity $\alpha(1) = \mathrm{id}$ to an arbitrary element $\alpha(0) = g \in \mathrm{Aut}^\Gamma(\widehat{X}_x, S)^0$.

   (2) Follows from (1). $\qquad\square$

Lemma 4.9.10. *Let $f : (X, S) \to (Y, R)$ be a Γ-smooth morphism of relative dimension k between Γ-stratified toroidal schemes. Let $x \in X^\Gamma$ be a closed K-rational point. Let Γ act trivially on $\mathbf{A}^k$ and $g_1, g_2 : X \to \mathbf{A}^k$ be any two Γ-equivariant morphisms such that $g_1(x) = g_2(x) = 0$ and $f \times g_i : (X, S) \to (Y, R) \times \mathbf{A}^k$ are Γ-étale for $i = 1, 2$. Then $f \times g_1$ and $f \times g_2$ determine the same orientation at x.*

Definition 4.9.11. We shall call such a morphism $f \times g$ an *étale extension* of f and denote it by $\widetilde{f}$.

Proof of 4.9.10. Let $y = f(x)$ and $y_1, \ldots, y_l$ be locally toric parameters at $y \in Y$. Let $x_1, \ldots, x_k$ be standard coordinates at $0 \in \mathbf{A}^k$. Set $v_i = f^*(y_i)$ for $i = 1, \ldots, l$ and $w_j^1 = g_1^*(x_j)$, $w_j^2 = g_2^*(x_j)$ for $j = 1, \ldots, k$. Then $v_1, \ldots, v_l, w_1^1, \ldots, w_k^1$ and $v_1, \ldots, v_l, w_1^2, \ldots, w_k^2$ are locally toric parameters at x. The automorphism $(\widehat{f \times g_1})_x \circ (\widehat{f \times g_2})_x^{-1} : \widehat{X}_x \to \widehat{X}_x$ maps w_i^1 to w_i^2. We can find a linear automorphism $\alpha_1 \in \{\mathrm{id}_v\} \times \mathrm{Gl}(k)$, which preserves v-coordinates and acts nontrivially on w_i^2-coordinates so that $\alpha_1^*(v_i) = v_i$ and $\alpha_1^*(w_i^2) = w_i^1 + z_i$, where z_i are some functions from the ideal $(w_1^2, \ldots, w_k^2)^2 + (v_1, \ldots, v_l)$. Clearly $\alpha_1 \in \mathrm{Aut}^\Gamma(\widehat{X}_x, S)^0$. Now consider the morphism $\mathbf{A}^1 \to \mathrm{Aut}^\Gamma(\widehat{X}_x, S)^0$ such that $\mathbf{A}^1 \ni t \mapsto \phi_t$, where

$\phi_t^*(v_i) = v_i$ and $\phi_t^*(w_j^1) = w_j^1 + t \cdot z_i$. This shows that $\phi_1 \in \operatorname{Aut}^\Gamma(\widehat{X}_x, S)^0$. Finally $(\widehat{f \times g_{1x}}) \circ (\widehat{f \times g_{2x}})^{-1} = \phi_1^{-1} \circ \alpha_1 \in \operatorname{Aut}^\Gamma(\widehat{X}_x, S)^0$. $\qquad \square$

Definition 4.9.12. We say that two Γ-smooth morphisms $f_1, f_2 : (X, S) \to (Y, R)$ of dimension k *determine the same orientation* at a closed K-rational point $x \in X^\Gamma$ if $f_1(x) = f_2(x)$ and there exist étale extensions $ef_1, \widetilde{f}_2 : (X, S) \to (Y, R) \times \mathbf{A}^k$ which determine the same orientation at x.

Lemma 4.9.13. *Let $f : (X, S) \to (Y, R)$ be a Γ-smooth morphism of Γ-stratified toroidal schemes over K. Let $x \in X^\Gamma$ and $y = f(x) \in Y^\Gamma$ be K-rational points and $s = f^{-1}(y) \in S$ be the stratum through x. Let $y_1, \ldots, y_l$ be locally toric Γ_x-semiinvariant parameters at y and $v_1 = f^*(y_1), \ldots, v_l = f^*(y_l), w_1, \ldots, w_k$ be locally toric Γ_x-semiinvariant parameters at x. Set $R := K[[w_1, \ldots, w_k]]$. Then $\widehat{\mathcal{O}}_{X,x} = R[[v_1, \ldots, v_l]]/I$, where I is the ideal span by binomial relations in v_i. $\operatorname{Aut}_R^\Gamma(\widehat{X}_x, S)$ is a proalgebraic group of Γ_x-equivariant automorphisms preserving strata and the subring R, and the monomorphism $\beta : \operatorname{Aut}_R^\Gamma(\widehat{X}_x, S) \to \operatorname{Aut}^\Gamma(\widehat{X}_x, S)$ induces an isomorphism*

$$\overline{\beta} : \operatorname{Aut}_R^\Gamma(\widehat{X}_x, S)/\operatorname{Aut}_R^\Gamma(\widehat{X}_x, S)^0 \to \operatorname{Aut}^\Gamma(\widehat{X}_x, S)/\operatorname{Aut}^\Gamma(\widehat{X}_x, S)^0.$$

Proof. It suffices to construct a surjective morphism $\alpha : \operatorname{Aut}^\Gamma(\widehat{X}_x, S) \to \operatorname{Aut}_R^\Gamma(\widehat{X}_x, S)$ with connected fibers such that $\alpha \circ \beta = \operatorname{id}_{\operatorname{Aut}_R^\Gamma(\widehat{X}_x, S)}$. Any automorphism ϕ from $\operatorname{Aut}^\Gamma(\widehat{X}_x, S)$ maps $w_1, \ldots, w_k, v_1, \ldots, v_l$ to $w_1', \ldots, w_k', v_1', \ldots, v_l'$. We put $\alpha(\phi)^*(w_i) = w_i, \alpha(\phi)^*(v_j) = v_j'$. Then $\alpha(\phi)$ is an endomorphism since w_i are algebraically independent of v_j. It is an automorphism since we can easily define an inverse homomorphism. The fiber $\alpha^{-1}(\alpha(\phi))$ consists of all elements of the type $\phi = \alpha(\phi) \circ \phi_1$, where $\phi_1^*(w_i) = w_i'$ and $\phi_1^*(v_j') = v_j'$. It is connected: for any two elements $\phi = \alpha(\phi) \circ \phi_1$ and $\phi' = \alpha(\phi) \circ \phi_1'$ we can define a rational map $\Phi : \mathbf{A}^1 \to \operatorname{Aut}^\Gamma(\widehat{X}_x, S)$ by $\Phi(t)^*(v_j') = v_j', \Phi(t)^*(w_i) = tw_i + (1-t)w_i'$. The latter is defined on an open subset $U \subset \mathbf{A}^1$ containing 0 and 1. $\qquad \square$

Lemma 4.9.14. *Let $\phi_i : (X, S) \to (Y, R)$ for $i = 1, 2$ be two Γ-smooth morphisms of Γ-stratified toroidal schemes such that $\phi_1(x) = \phi_2(x) = y \in Y$ for a K-rational point $x \in X^\Gamma$ and strata in S are preimages of strata in T and $\phi_1^{-1}(y) = \phi_2^{-1}(y)$. Assume there exists a smooth scheme V with a trival action of Γ and a Γ-equivariant morphism $g : X \to V$ such that $\phi_i \times g : (X, S) \to (Y, R) \times V$ are Γ-smooth. Define $X' := g^{-1}(g(x)), S' := \{s \cap X' \mid s \in S\}$.*

Then ϕ_1 and ϕ_2 determine the same orientation iff their restrictions $\phi_i' : (X', S') \to (Y, R)$ do.

Proof. We can assume that g is Γ-étale by replacing, if necessary, (Y, R) with $(Y, R) \times \mathbf{A}^m$, where $m = \dim(X) - \dim(Y) - \dim(V)$, and $\phi_i : (X, S) \to (Y, R)$ and $\phi_i \times g : (X, S) \to (Y, R) \times V$ with Γ-smooth morphisms $\widetilde{\phi}_i : (X, S) \to (Y, R) \times \mathbf{A}^m$ and Γ-étale morphisms $\widetilde{\phi}_i \times g : (X, S) \to (Y, R) \times \mathbf{A}^m \times V$.

Let $y = f(x)$ and $y_1, \ldots, y_l$ be locally toric Γ-semiinvariant parameters at $y \in Y$. Let $x_1, \ldots, x_k$ be locally toric parameters at $p := g(x) \in V$. Set $v_i^1 = \phi_1^*(y_i), v_i^2 = \phi_2^*(y_i)$ for $i = 1, \ldots, l$ and $w_j = g^*(x_j)$ for $j = 1, \ldots, k$. Then $v_1^1, \ldots, v_l^1, w_1, \ldots, w_k$ and $v_1^2, \ldots, v_l^2, w_1, \ldots, w_k$ are locally toric parameters at x. The automorphism $\alpha := (\widehat{\phi_1 \times g})_x^{-1} \circ (\widehat{\phi_2 \times g})_x : \widehat{X}_x \to \widehat{X}_x$ maps the first set of parameters onto the second one. Thus α belongs to the group of automorphisms $\operatorname{Aut}_R^\Gamma(\widehat{X}_x, S)$ preserving $R = K[[w_1, \ldots, w_k]]$. The restriction of each automorphism from $\operatorname{Aut}_R^\Gamma(\widehat{X}_x, S)$ to $X' = \operatorname{Spec}(\widehat{\mathcal{O}}_{X,x}/(w_1, \ldots, w_k))$ is an automorphism. On the other hand we can write $\widehat{\mathcal{O}}_{X,x} = \widehat{\mathcal{O}}_{X',x}[[w_1, \ldots, w_k]]$. Hence each automorphism in $\operatorname{Aut}_K^\Gamma(\widehat{X'}_x, S')$ determines an automorphism in $\operatorname{Aut}_R^\Gamma(\widehat{X}_x, S)$. We come to a natural epimorphism of proalgebraic groups:

$$\operatorname{res} : \operatorname{Aut}_R^\Gamma(\widehat{X}_x, S) \to \operatorname{Aut}_K^\Gamma(\widehat{X'}_x, S').$$

The kernel of res is a proalgebraic group H consisting of all automorphisms $\beta \in \operatorname{Aut}_R^\Gamma(\widehat{X}_x, S)$ which can be written in the form $\beta(v_i^1) = v_i^1 + r_i, \beta(w_j) = w_j$, where $r_i \in (w_1, \ldots, w_k) \cdot \widehat{\mathcal{O}}_{X,x}$. For any fixed β and $t \in \mathbf{A}^1$ yield $r_i^t := r_i(v_1^1, \ldots, v_l^1, t \cdot w_1, \ldots, t \cdot w_k)$ and $\beta^t(v_i^1) = v_i^1 + r_i^t, \beta^t(w_j) = w_j$. This gives a morphism $t : \mathbf{A}^1 \to H$ such that

$t(1) = \beta^1 = \beta$ and $t(0) = \operatorname{id}$. Consequently, H is connected and

$$res^{-1}(\operatorname{Aut}_K^\Gamma(\widehat{X'}_x, S')^0 = H \cdot (\operatorname{Aut}_K^\Gamma(\widehat{X'}_x, S')^0 = \operatorname{Aut}_R^\Gamma(\widehat{X}_x, S)^0.$$

By the above and Lemma 4.9.13 the homomorphisms

$$\mathrm{Aut}_K^\Gamma(\widehat{X}_x, S) \xleftarrow{\ \beta\ } \mathrm{Aut}_R^\Gamma(\widehat{X}_x, S) \xrightarrow{\ \mathrm{res}\ } \mathrm{Aut}_K^\Gamma(\widehat{X'}_x, S')$$

define isomorphisms

$$\mathrm{Aut}_K^\Gamma(\widehat{X}_x, S)/\mathrm{Aut}_K^\Gamma(\widehat{X}_x, S)^0 \xrightarrow{\ \overline{\beta}^{-1}\ } \mathrm{Aut}_R^\Gamma(\widehat{X}_x, S)/\mathrm{Aut}_R^\Gamma(\widehat{X}_x, S)^0 \xrightarrow{\ \overline{\mathrm{res}_1}\ } \mathrm{Aut}_K^\Gamma(\widehat{X'}_x, S')/\mathrm{Aut}_K^\Gamma(\widehat{X'}_x, S')^0.$$

Finally we see that $(\widehat{\phi_1 \times g})_x \circ (\widehat{\phi_2 \times g})_x^{-1} \in \mathrm{Aut}_K^\Gamma(\widehat{X}_x, S)^0$ iff $\widehat{\phi'_1} \circ \widehat{\phi'}_{2x}^{-1} \in \mathrm{Aut}_K^\Gamma(\widehat{X'}_x, S')^0$ $\square$

4.10. **Orientation on stratified toroidal varieties.**

Definition 4.10.1. Let (X, S) be a Γ-stratified toroidal variety (or Γ-stratified toroidal scheme) with an associated Γ-semicomplex Σ. Let $\tau \leq \sigma$ and $x \in s$. Let $\imath_\tau^\sigma : \tau \to \sigma$ denote the standard embedding and $\phi_\sigma : U_\sigma \to X_\sigma$ be a chart. For any $\sigma' \preceq \sigma$ in N_σ such that $\omega(\sigma') = \tau$ in N_τ we denote by $\phi_\sigma^{\sigma'} : U_\sigma^{\sigma'} := \phi_\sigma^{-1}(X_{\sigma'}) \to X_{\sigma'}$ the restriction of ϕ_σ to $U_\sigma^{\sigma'}$.

Let $\pi_{\sigma'}^\tau : X_{\sigma'} \to X_\tau$ denote the toric morphism induced by any projection $\overline{\pi}_{\sigma'}^\tau : \sigma' \to \tau$ such that $\overline{\pi}_{\sigma'}^\tau \circ \imath_\tau^\sigma = \mathrm{id}_{|\tau}$.

1. We say that the Γ-stratified toroidal variety (X, S) with atlas $\mathcal{U}$ is *oriented* if for any two charts $\phi_i : U_i \to X_{\sigma_i}$, where $i = 1, 2$, and any $\sigma_i' \preceq \sigma_i$ such that $\omega(\sigma_i') = \sigma \leq \sigma_i$ the Γ_σ-smooth morphisms $\pi_{\sigma'}^\sigma \phi_{\sigma_i}^{\sigma_i'} : U_{\sigma_i}^{\sigma_i'} \to X_\sigma$ determine the same orientation at any $x \in U_{\sigma_1}^{\sigma_1'} \cap U_{\sigma_2}^{\sigma_2'} \cap \mathrm{strat}_X(\sigma)$.
2. Let (X, S) be a Γ-stratified toroidal scheme with atlas $\mathcal{U}$ which contains a reduced subscheme W of finite type over K. We say that (X, S) is *oriented* along W if for any two charts $\phi_i : U_i \to X_{\sigma_i}$, where $i = 1, 2$, and any $\sigma_i' \preceq \sigma_i$ such that $\omega(\sigma_i') = \sigma \leq \sigma_i$ the Γ_σ-smooth morphisms $\pi_{\sigma_i'}\phi_{\sigma_i}^{\sigma_i'} : U_{\sigma_i}^{\sigma_i'} \to X_\sigma$ determine the same orientation at any $x \in U_{\sigma_1}^{\sigma_1'} \cap U_{\sigma_2}^{\sigma_2'} \cap \mathrm{strat}_X(\sigma) \cap W$.
3. We shall call such W a *K-subscheme*.

The following lemma is a generalization of Lemma 4.8.3

Lemma 4.10.2. *Let (X_Σ, S) be a Γ-stratified toric variety corresponding to an embedded semifan $\Omega \subset \Sigma$. Then (X_Σ, S) is an oriented Γ-stratified toroidal variety with the associated oriented Γ-semicomplex Ω^{semic} and atlas $\mathcal{U}^{\mathrm{can}}(\Omega, \Sigma)$.* $\square$

Remark. By Lemma 4.9.10 the above definition does not depend upon the choice of the projection π_σ.

Definition 4.10.3. Let (X, S) be a Γ-stratified toroidal variety (respectively a Γ-stratified toroidal scheme with a K-subscheme W) with two atlases $\mathcal{U}_1$ and $\mathcal{U}_2$ such that $(X, S, \mathcal{U}_1)$ and $(X, S, \mathcal{U}_2)$ are oriented (resp. oriented along W). Then $\mathcal{U}_1$ and $\mathcal{U}_2$ on (X, S) are *compatible* (resp. *compatible along W*) if $(X, S, \mathcal{U})$, where $\mathcal{U} := \mathcal{U}_1 \cup \mathcal{U}_2$, is oriented (resp. oriented along W).

Lemma 4.10.4. *Let $f : (X, S) \to (Y, R)$ be a Γ-smooth morphism of Γ-stratified toroidal schemes such that the strata in S are preimages of strata in R and all strata in R are dominated by strata in S. Assume that $(Y, R, \mathcal{U})$ with associated Γ-semicomplex Σ is oriented along a K-subscheme W. Define $f^*(\mathcal{U}) := \{\phi f \mid \phi \in \mathcal{U}\}$. Then $(X, S, f^*(\mathcal{U}))$ with the associated Γ-semicomplex Σ is oriented along any K-subscheme $W' \subset f^{-1}(W)$.*

Proof. Let $\phi : U \to X_\sigma$ be a chart on Y. Then ϕ is Γ-equivariant, Γ_σ-smooth morphism such that the intersections $r \cap U$ are inverse images of strata of X_σ. Then $\phi f : \phi^{-1}(U) \to X_\sigma$ has exactly the same properties. Since $X_\sigma // \Gamma_\sigma$ exists it follows from Lemma 4.1.3 that ϕf is Γ_σ-smooth. $\square$

Lemma 4.10.5. *Let $\phi_i : (X, S) \to (Y, R)$ for $i = 1, 2$ be two Γ-smooth morphisms of Γ-stratified toroidal schemes such that the strata in S are preimages of strata in R. Assume $(Y, R, \mathcal{U})$ is oriented and there exists a Γ-equivariant morphism $g : X \to V$ into a smooth scheme with trivial action of Γ, such that $\phi_i \times g : (X, S) \to (Y, R) \times V$ are Γ-smooth. Set $X' := g^{-1}(g(x)), S' := S \cap X'$. Let $W \subset X'$ be a K-subscheme. Let $\psi_i : (X', S') \to (Y, R)$ denote the restrictions of ψ_i.*

Then $\phi_1^*(\mathcal{U})$ and $\phi_2^*(\mathcal{U})$ are compatible on (X, S) along W iff $\psi_1^*(\mathcal{U})$ and $\psi_2^*(\mathcal{U})$ are compatible on (X', S') along W.

Proof. Follows from Lemma 4.9.14. $\square$

Example 4.10.6. *The group of automorphisms of the formal completion of a toroidal embedding at a closed point is connected.*

The formal completion of a toroidal embedding at a closed point is isomorphic to $\widehat{X}_{\sigma \times \mathrm{reg}(\sigma)}$, where $\mathrm{reg}(\sigma)$ is a regular cone and strata are defined by all faces of σ. Let $x_1, \dots, x_k$ be semiinvariant coordinates on $\widehat{X}_\sigma$ corresponding to characters $m_1, \dots, m_k$. Denote by $x_{k+1}, \dots, z_{k+l}$ the standard coordinates on $X_{\mathrm{reg}(\sigma)}$ corresponfing to characters $m_{k+1}, \dots, m_{k+l}$. Let ϕ be an automorphism of $\widehat{X}_{\sigma \times \mathrm{reg}(\sigma)}$ which preserves the torus orbit stratification. Since all the divisors in the orbit stratification are preserved, the divisors of x_i, where $i = 1, \dots, k$, are also preserved, so we have $\phi^*(x_i) = x_i s_i$, where $s_i = a_{i0} + a_{i\alpha} x^\alpha + \dots$ is invertible and $\alpha = (\alpha_1, \dots, \alpha_{k+l})$ denote multiindices. By Lemma 4.9.10, we may assume that $\phi^*(x_i) = x_i$ for $i = k+1, \dots, l$. An integral vector $v := (v_1, \dots, v_{l+k}) \in \mathrm{int}(\sigma \times \mathrm{reg}(\sigma))$ defines the 1-parameter subgroup $t \mapsto t^v$ of the "big" torus T. For any $t \in T$ let t^v denote the automorphism defined as

$$x_i \mapsto x_i \cdot t^{\langle v, m_i \rangle}.$$

consider the morphism $\phi_t := t^{-v} \phi t^v$. In particular $\phi_1 = \phi$ and

$$x_i \circ \phi_t = x_i (a_{i0} + t^{\alpha_1 m_1 v_1 + \dots + \alpha_{l+k} m_{l+k} v_{l+k}} a_{i\alpha} x^\alpha + \dots),$$

where $\alpha_1 m_1 v_1 + \dots + \alpha_{l+k} m_{l+k} v_{l+k} = \langle v, \alpha_1 m_1 + \dots + \alpha_{l+k} m_{l+k} \rangle \geq 0$. Then ϕ_0 is the well defined automorphism $x_i \mapsto a_{i0} x_i$. The latter automorphism belongs to the torus T. Since we can "connect" any automorphism ϕ with an element of the torus the connectedness of the group of automorphisms follows.

Any toroidal embedding is an oriented stratified variety.

By Lemma 4.9.14 and by the connectedness of the automorphism group any two charts determine the same orientation.

Example 4.10.7. Let $X := \{x \in \mathbf{A}^4 \mid x_1 x_2 = x_3 x_4\}$ be a toric variety with isolated singularity. Then X is a stratified toroidal variety with the stratification consisting of the singular point p and of its complement. The blow-up of the ideal of the point is a resolution of singularities with the exceptional divisor determining the valuation. Consequently the valuation determined by the point is preserved by all automorphisms in $\mathrm{Aut}(\widehat{X}_x, S)$. We have the natural homomorphism $d : \mathrm{Aut}(\widehat{X}_x, S) \to \mathrm{Gl}(\mathrm{Tan}_{X,x})$. The kernel of d is a connected proalgebraic group (see Lemma 7.3.1(3)). The image $d(\mathrm{Aut}(\widehat{X}_x, S))$ consists of linear automorphisms preserving the ideal of $x_1 x_2 - x_3 x_4$, that is, multiplying the polynomial by a nonzero constant. Hence $d(\mathrm{Aut}(\widehat{X}_x, S)) = K^* \cdot O$, where K^* acts by multiplying the coordinates by $t \in K^*$ and O is the group of linear automorphisms preserving $x_1 x_2 - x_3 x_4$. By the linear change of coordinates $x_1 = y_1 - iy_2$, $x_2 = y_1 + iy_2$, $x_3 = y_3 - iy_4$, $x_4 = y_3 + iy_4$ we transform the polynomial into $y_1^2 + y_2^2 + y_3^2 + y_4^2$ ($\mathrm{char}(K) \neq 2$). This shows that O is conjugate to the group of orthogonal matrices with K-rational coefficients $O(4, K)$. $O(4, K)$ consists of two components with $O(4, K)^0 = SO(4, K)$. Therefore $\mathrm{Aut}(\widehat{X}_x, S) = ker(d) \cdot K^* \cdot O$ consists of two components. Fix any isomorphism $\phi : X \to X_\sigma$, which can be considered as a chart. An oriented atlas consists of charts compatible with ϕ. Consequently, there are two orientations corresponding to any two incompatible isomorphisms $\phi : X \to X_\sigma$.

4.11. **Subdivisions of oriented semicomplexes.**

Definition 4.11.1. For any Γ-semicone σ denote by $\mathrm{Aut}(\sigma)^0$ the group of all its automorphisms which define the Γ-equivariant automorphisms of X_σ preserving orientation.

By an *oriented semicomplex* (resp. *oriented Γ-semicomplex*) we mean a semicomplex (resp. Γ-semicomplex) Σ for which for any $\sigma \leq \tau \leq \gamma$ there is $\alpha_\sigma \in \mathrm{Aut}(\sigma)^0$ for which $\imath_\tau^\gamma \imath_\sigma^\tau = \imath_\sigma^\gamma \alpha_\sigma$.

Definition 4.11.2. By an *isomorphism* of two oriented Γ-semicomplexes $\Sigma \to \Sigma'$ we mean a bijection of faces $\Sigma \ni \sigma \mapsto \sigma' \in \Sigma'$ along with a collection of face isomorphisms $j_\sigma : \sigma \to \sigma'$ such that for any $\tau \leq \sigma$, there is an automorphism $\beta_\tau \in \mathrm{Aut}(\tau)^0$ such that $j_\sigma \imath_\tau^\sigma = \imath_{\tau'}^{\sigma'} j_\tau \beta_\tau$.

Remarks. 1. An oriented semicomplex can be viewed as an oriented Γ-semicomplex with trivial groups Γ_σ.
2. We will show that the Γ-semicomplex associated to an oriented Γ-stratified toroidal variety is oriented (Lemma 8.2.5).
3. The description of the group $\mathrm{Aut}(\sigma)^0$ is given in Lemma 8.1.2.

Lemma 4.11.3. *Let Δ be a subdivision of a fan Σ and $\sigma \in \Sigma$. Then $\Delta|\sigma := \{\delta \in \Delta \mid \delta \subset \sigma\}$ is a subdivision of σ.* $\qquad\square$

Definition 4.11.4. A *subdivision* of an oriented Γ-semicomplex Σ is a collection Δ of fans Δ^σ in N_σ where $\sigma \in \Sigma$ such that

1. For any $\sigma \in \Sigma$, Δ^σ is a subdivision of $\overline{\sigma}$ which is $\mathrm{Aut}(\sigma)^0$-invariant.
2. For any $\tau \leq \sigma$, $\Delta^\sigma|\overline{\tau} = \imath^\sigma_\tau(\Delta^\tau)$.

Remarks.   1. By abuse of terminology we shall understand by a subdivision of a semicone σ a subdivision of the fan $\overline{\sigma}$.
2. A subdivision of an oriented semicomplex can be viewed as a subdivision of an oriented Γ-semicomplex with trivial groups Γ_σ.
3. By definition vectors in faces σ of an oriented Γ-semicomplex Σ are defined up to automorphisms from $\mathrm{Aut}(\sigma)^0$. Consequently, the faces of subdivisions Δ^σ are defined up to automorphisms from $\mathrm{Aut}(\sigma)^0$ and in general don't give a structure of semicomplex. Only invariant faces of Δ^σ may define a semicomplex.
4. The condition on Δ^σ to be $\mathrm{Aut}(\sigma)^0$-invariant is for *canonical* subdivisions replaced with a somewhat stronger condition of similar nature which says that the induced morphism $\widehat{X}_{\Delta^\sigma} := X_{\Delta^\sigma} \times_{X_\sigma} \widehat{X}_\sigma \to \widehat{X}_\sigma$ is $\mathrm{Aut}^\Gamma(\widehat{X}_\sigma)^0$-equivariant. The latter condition can be translated into the condition that all "new" rays of the subdivision are in the *stable support* of Σ.

Definition 4.11.5. Let Δ be a subdivision of an oriented Γ-semicomplex Σ. Let ϱ_σ be an ray passing through the interior of the cone $\sigma \in \Sigma$ such that defining a collection of rays $\varrho := \{\varrho_\tau \in \tau \mid \tau \geq \sigma\}$ such that for any $\tau \geq \sigma$, the ray $\varrho_\tau = \imath^\tau_\sigma(\varrho_\sigma)$ is $\mathrm{Aut}(\tau)^0$-invariant . By the *star subdivision* of Δ at ρ we mean the subdivision

$$\varrho \cdot \Delta := \{\varrho_\tau \cdot \Delta^\tau \mid \tau \geq \sigma\} \cup \{\Delta^\tau \mid \tau \not\geq \sigma\}.$$

4.12. **Toroidal morphisms of stratified toroidal varieties.**

Definition 4.12.1. Let (X, S) be a a stratified toroidal variety (resp. a Γ-stratified toroidal variety) with an atlas $\mathcal{U}$. We say that Y is a *toroidal modification* of (X, S) if

1. There is given a proper morphism $f : Y \to X$ (resp. proper Γ-equivariant morphism) such that for any $x \in s = \mathrm{strat}_X(\sigma)$ there exists a chart $x \in U_\sigma \to X_\sigma$, a subdivision Δ^σ of σ, and a fiber square

$$
\begin{array}{ccc}
U_\sigma & \xrightarrow{\phi_\sigma} & X_\sigma \\
\uparrow f & & \uparrow \\
U_\sigma \times_{X_\sigma} X_{\Delta^\sigma} \quad \simeq \quad f^{-1}(U_\sigma) & \xrightarrow{\phi^f_\sigma} & X_{\Delta^\sigma}
\end{array}
$$

2. For any point x in a stratum s every automorphism (resp. Γ_s-equivariant automorphism) α of $\widehat{X}_x$ preserving strata and orientation can be lifted to an automorphism (resp. Γ_s-equivaiant automorphism) α' of $Y \times_X \widehat{X}_x$.

Definition 4.12.2. Let Y be a toroidal modification of a stratified toroidal variety (X, S). By a *canonical stratification* R on Y we mean the finest stratification on Y satisfying the conditions:

1. Strata of R are mapped onto strata of S.
2. For any chart $\phi_\sigma : U_\sigma \to X_\sigma$ on (X, S) there is an embedded subdivision $\Omega^\sigma \subset \Delta^\sigma$ such that the strata of $R \cap f^{-1}(U_\sigma)$ are defined as inverse images $(\phi^f_\sigma)^{-1}(\mathrm{strat}(\omega))$ of strata on $(X_{\Delta^\sigma}, S_{\Omega^\sigma})$.
3. For any point x in a stratum s every Γ_s-equivariant automorphism α of $\widehat{X}_x$ preserving strata and orientation can be lifted to an Γ_s-equivariant automorphism α' of $Y \times_X \widehat{X}_x$ preserving strata.

If Y is a toroidal modification of (X, S) and R is a canonical stratification on Y then we shall also speak of a *toroidal morphism* $(Y, R) \to (X, S)$ of stratified toroidal varieties.

Remarks.   1. The definition of a toroidal morphism of stratified toroidal varieties is a generalization of the definition of a toroidal morphism of toroidal embeddings.

2. The condition (2) of Definition 4.12.1 is similar to Hironaka's "allowability" condition, used by Mumford in the theory of toroidal embeddings (see [39] and section 1.1). It means that a toroidal morphism, which by condition (1) is induced locally by charts, in fact does not depend on the charts i.e., if a morhism which satisfies the Hironaka condition is induced locally by the diagram (1) for some chart, then it is also induced locally by any other chart and the given subdivision (see Lemma 4.12.11).

3. In the above definition Ω^σ consists of those faces of Δ^σ whose relative interior intersect the *stable support* (see Lemma 6.5.1).

4. Only some special subdivisions Δ^σ of σ define (locally) toroidal modifications (see example 4.12.3). These subdivisions, called *canonical*, induce modifications which are independent of the charts. They are characterized by the property that "new" rays of the subdivision are contained in the stable support (Proposition 7.6.1).

Example 4.12.3. Let $X = \mathbf{A}^2$ and S be the stratification on X that consists of the point $s_0 = (0,0)$ and its complement $s_1 = \mathbf{A}^2 \setminus \{(0,0)\}$. Then (X, S) is an oriented stratified toroidal variety corresponding to the semicomplex $\Sigma = \{\sigma_0, \sigma_1\}$ where $\sigma_0 = \langle (1,0), (0,1) \rangle$, $\sigma_1 = \{(0,0)\}$.

- Let $Y \to X$ be a toric morphism corresponding to the normalized blow-up of $I = (x^2, y)$. This morphism of toric varieties corresponds to the subdivision $\Delta := \langle (1,2) \rangle \cdot \Sigma$ of Σ. Therefore condition (1) of Definition 4.12.1 is satisfied for the identity chart.

  The condition (2) of the definition is not satisfied. The automorphism α defined by $\alpha(x) = y$, $\alpha(y) = x$ doesn't lift.

- Let $Y \to X$ be a blow-up of $I = (x, y)$. Then Y is a toric variety and let S_Y be the stratification consisting of s_1 and the toric orbits in the exceptional curve $\mathbf{P}^1$: $s_2 = \{0\}$, $s_3 = \{\infty\}$ and $s_4 = \mathbf{P}^1 \setminus \{0\} \setminus \{\infty\}$. Then Y is a toroidal <u>modification</u> of (X, S). The automorphism α defined by $\alpha(x) = y$, $\alpha(y) = x$ lifts to the automorphism permuting s_2 and s_3. Therefore $(Y, S_Y) \to (X, S)$ is NOT a toroidal <u>morphism</u>.

- Let $Y \to X$ be a blow-up of $I = (x, y)$ and S_Y be the stratification consisting of the exceptional curve $s_2 = D$ and its complement s_1. Then $(Y, S_Y) \to (X, S)$ is a toroidal morphism corresponding to the subdivision Δ of Σ, where $\Delta = \{\langle (1,0), (1,1) \rangle, \langle (1,1) \rangle, \langle (1,1), (0,1) \rangle, \{(0,0)\}$. The stratification S_Y is described by the semicomplex $\Omega = \{\langle (1,1) \rangle, \{(0,0)\}\} \subset \Delta$. After blow-up some faces "disappear" from $\Omega \neq \Delta$. The remaining faces of Ω are $\mathrm{Aut}(\sigma)^0$-invariant.

Lemma 4.12.4. *Let $Y \to X$ be a toroidal modification of an oriented Γ-stratified toroidal variety (X, S). Then all fibers $f^{-1}(x)$, where $x \in s$, are isomorphic.*

Proof. It follows from condition (1) of Definition 4.12.1 that all fibers $f^{-1}(x)$, where $x \in s = \mathrm{strat}_X(\sigma) \cap U_\sigma$, are isomorphic for any chart $U_\sigma \to X_\sigma$. $\qquad\square$

Lemma 4.12.5. *Let $Y \to X$ be a toroidal modification of an oriented Γ-stratified toroidal variety (X, S). Then the exceptional divisors of the morphism $f : Y \to X$ dominate strata.*

Proof. Suppose an exceptional divisor D does not dominate a stratum. The generic point of $f(D)$ is in a stratum s and $\dim(f(D)) < \dim(s)$. By Lemma 4.12.4, s is not a generic open stratum in X. The dimension of a generic fiber $f^{-1}(x)$, where $x \in f(D)$, is greater than or equal to $\dim(D) - \dim(f(D)) = n - 1 - \dim(f(D))$. The dimension of a generic fiber $f^{-1}(x)$, where $x \in s$, is at most $\dim(f^{-1}(s)) - \dim(s) < n - 1 - \dim(f(D))$. This contradicts Lemma 4.12.4. $\qquad\square$

Definition 4.12.6. If Σ is a fan then we denote by $\mathrm{Vert}(\Sigma)$ the set of all 1-dimensional faces (rays) in Σ.

Lemma 4.12.7. *Let $\sigma = \tau_1 \oplus \tau_2$ and Δ be a subdivision of τ_1. Then*
$\Delta \oplus \tau_2 := \{\delta \oplus \sigma \mid \delta \in \Delta, \ \sigma \preceq \tau_2\}$ *is a subdivision of τ called the <u>joint</u> of Δ and τ_2.* $\qquad\square$

Lemma 4.12.8. *Let $\Omega \subset \Sigma$ be an embedded Γ-semifan. Let Δ be a subdivision of Σ such that any ray $\varrho \in \mathrm{Vert}(\Delta \setminus \Sigma)$ passes through the interior of $\omega \in \Omega$. Then for any cone $\sigma = \omega(\sigma) \oplus \mathrm{r}(\sigma)$ in Σ, $\Delta|\sigma = (\Delta|\omega(\sigma)) \oplus \mathrm{r}(\sigma)$.*

Proof. Since $\omega := \omega(\sigma)$ is a maximal face of σ which belongs to Ω, all "new" rays which are contained in σ are in ω and therefore the maximal cones in $\Delta|\sigma$ are of the form $\tau = \omega \oplus \mathrm{r}(\sigma)$ and $\tau \subset \omega$. This shows that $\Delta_1 := \Delta|\sigma$ is a subfan of $\Delta_2 := (\Delta|\omega) \oplus \mathrm{r}(\sigma)$. By properness we get $|\Delta_1| = |\Delta_2|$ and hence $\Delta_1 = \Delta_2$. $\qquad\square$

Lemma 4.12.9. . *Let $\psi_1 : Y_1 \to X$ and $\psi_2 : Y_2 \to X$ be proper birational morphisms of normal noetherian schemes.*

1. *Suppose $\widehat{X}_x \times_X Y_1$ and $\widehat{X}_x \times_X Y_2$ are isomorphic over $\widehat{X}_x$ for a K-rational point $x \in X$. Then there exists an open neighborhood U of x such that the proper birational map $\psi_1^{-1}(U) \to \psi_2^{-1}(U)$ is an isomorphism.*
2. *Suppose $\widehat{X}_x \times_X Y_1 \to \widehat{X}_x \times_X Y_2$ is a proper morphism over $\widehat{X}_x$ for a K-rational point $x \in X$. Then there exists an open neighborhood U of x such that the proper birational map $\psi_1^{-1}(U) \to \psi_2^{-1}(U)$ is a morphism.*

Proof (1) Suppose that the sets $\psi_1^{-1}(U)$ and $\psi_2^{-1}(U)$ are not isomorphic for any U. Then the closed subset of Y_2 where $\psi_1\psi_2^{-1}$ is not an isomorphism interesect the fiber $\psi_2^{-1}(x)$.

Let Y_3 be a component in $Y_1 \times_X Y_2$ which dominates X. Since all varieties are normal, at least one of the proper morphisms $Y_3 \to Y_i$ is not an isomorphism for $i = 1, 2$ over $\psi_i^{-1}(U)$ for any $U \ni x$. Then it collapses a curve to a point over x.

Let $\widehat{X}_3$ be a component in $\widehat{X}_1 \times_{\widehat{X}_x} \widehat{X}_2$ which dominates $\widehat{X}_x$. The morphisms $\widehat{X}_3 \to \widehat{X}_i$ for $i = 1, 2$ are pull-backs of the morphisms $Y_3 \to Y_i$ via étale morphisms. This shows that they are not both isomorhisms, and neither is the induced birational map $\widehat{X}_1 \to \widehat{X}_2$, which contradicts the assumption.

(2) The same reasoning.   □

Lemma 4.12.10. *Let $\phi_\sigma : U_\sigma \to X_\sigma$ denote a chart on an oriented Γ-stratified toroidal variety (X, S). Let $\sigma' \preceq \sigma$ denote a face for which $\omega(\sigma') = \varrho \leq \sigma$.*

Consider the smooth morphism $\psi_\sigma := \pi_{\sigma'}^\varrho \phi_\sigma^{\sigma'} : U_{\sigma'} \to X_\varrho$ induced by the chart ϕ_σ.

Let Δ^σ be a subdivision of σ and $\Delta^\varrho := \Delta^\sigma|_\varrho$ be a subdivision of ϱ.

Suppose $f_\sigma : \widetilde{U}_\sigma := U_\sigma \times_{X_\sigma} X_{\Delta^\sigma} \to U_\sigma$ satisfies Condition (2) of Definition 4.12.1. Then the following diagram consists of two fiber squares of smooth morphisms:

$$
\begin{array}{ccccc}
 & U_\sigma^{\sigma'} & \longrightarrow & X_{\sigma'} = X_{\varrho \oplus \mathrm{r}(\sigma')} & \longrightarrow & X_\varrho \\
 & \uparrow f & & \uparrow & & \uparrow \\
U_\sigma := U \times_{X_{\sigma'}} X_{\Delta^\sigma} \quad \simeq & f^{-1}(U_\sigma^{\sigma'}) & \to & X_{\Delta^\sigma|\sigma'} = X_{\imath_\varrho^\sigma(\Delta^\varrho) \oplus \mathrm{r}(\sigma')} & \to & X_{\Delta^\varrho}
\end{array}
$$

Proof It follows from Lemma 4.12.5 that the assumption of Lemma 4.12.8 is satisfied for Δ^σ. By Lemma 4.12.8, $\Delta^\sigma|\sigma' = \Delta^\sigma|_\varrho \oplus \mathrm{r}(\sigma') = \imath_\varrho^\sigma(\Delta^\varrho) \oplus \mathrm{r}(\sigma')$.   □

Lemma 4.12.11. *Let $\phi_\sigma : U_\sigma \to X_\sigma$ and $\phi_\tau : U_\tau \to X_\tau$ denote two charts on an oriented Γ-stratified toroidal variety (X, S). Let Δ^σ be a subdivision of σ and Δ^τ denote a subdivision of τ. Assume that for any $\varrho \leq \sigma, \tau$, $\Delta^\sigma|\varrho = \Delta^\tau|\varrho$. Suppose $f_\sigma : \widetilde{U}_\sigma := U_\sigma \times_{X_\sigma} X_{\Delta^\sigma} \to U_\sigma$ and $f_\tau : \widetilde{U}_\tau := U_\tau \times_{X_\tau} X_{\Delta^\tau} \to U_\tau$ satisfy Condition (2) of Definition 4.12.1.*

Then $\phi_\sigma^{-1}(U)$ and $\phi_\tau^{-1}(U)$ are isomorphic over $U := U_\sigma \cap U_\tau$.

Proof. Suppose that the relevant sets are not isomorphic over $\mathrm{Spec}(\mathcal{O}_x)$ for $x \in \mathrm{strat}_X(\varrho) \cap U$. Then there are faces $\sigma' \preceq \sigma$ and $\tau' \preceq \tau$ such that $\omega(\sigma') = \omega(\tau') = \varrho$.

Set by $\mathrm{reg}(\varrho) = \langle e_1, \ldots, e_{\dim(\mathrm{strat}_X(\varrho))} \rangle$. The diagram from Lemma 4.12.10 gives the following fiber square diagram of étale extensions:

$$
\begin{array}{ccc}
U & \longrightarrow & X_{\varrho \times reg(\varrho)} \\
\uparrow f_\sigma & & \uparrow \\
f_\sigma^{-1}(U) & \longrightarrow & X_{\Delta \times reg(\varrho)}
\end{array}
\qquad
\begin{array}{ccc}
U & \longrightarrow & X_{\varrho \times reg(\varrho)} \\
\uparrow f_\tau & & \uparrow \\
f_\tau^{-1}(U) & \to & X_{\Delta \times reg(\varrho)}
\end{array}
$$

where the horizontal morphisms are étale.

Consider the following fiber square diagram:

$$
\begin{array}{ccccc}
\widehat{X} \times_U U_\sigma & \xrightarrow{\alpha'} & \widehat{X} \times_U U_\tau & \xrightarrow{\widehat{\psi}_\sigma'} & X_{\Delta^\varrho \times \mathrm{reg}(\varrho)} \times_{X_{\varrho \times \mathrm{reg}(\varrho)}} \widehat{X}_{\varrho \times \mathrm{reg}(\varrho)} \\
\downarrow & & \downarrow & & \downarrow \\
\widehat{X}_x & \xrightarrow{\alpha} & \widehat{X}_x & \xrightarrow{\widehat{\psi}_\sigma} & \widehat{X}_{\varrho \times \mathrm{reg}(\varrho)},
\end{array}
$$

where $\widehat{\psi}_\sigma$ and $\widehat{\psi}_\tau$ are isomorphisms of the completions of the local schemes induced by étale extensions of the smooth morphisms ψ_σ and ψ_τ, α is an isomorphism such that $\widehat{\psi}_\tau := \widehat{\psi}_\sigma \circ \alpha$, and $\widehat{\psi}'_\sigma$, $\widehat{\psi}'_\tau$ and α' are liftings of $\widehat{\psi}_\sigma$, $\widehat{\psi}_\tau$ and α.

It follows from the diagram that α is an automorphism of $\widehat{X}_x$ which can be lifted to an automorphism α' of $\widehat{X}_1$. Note that α preserves strata and orientation. It follows that $\widehat{X} \times_U U_\sigma$ and $\widehat{X} \times_U U_\tau$. are isomorphic. By Lemma 4.12.9, $\widetilde{U}_\sigma$ and $\widetilde{U}_\tau$ are isomorphic over a neighborhoood of $x \in X$, which contradicts the choice of $x \in X$. $\square$

4.13. **Canonical subdivisions of oriented semicomplexes.** As a corollary from the proof of Lemma 4.12.11 we obtain the following lemma.

Lemma 4.13.1. *Let (X, S) be an oriented Γ-stratified toroidal variety of dimension n with associated oriented Γ-semicomplex Σ and let $f : Y \to (X, S)$ be a toroidal modification. Let x be a closed point in stratum $\mathrm{strat}_X(\sigma) \in S$, $\phi_\sigma : U \to X_\sigma$ be a chart of a neighborhood U of x and Δ^σ be a subdivision of σ for which there is a fiber square*

$$
\begin{array}{ccc}
U & \xrightarrow{\phi_\sigma} & X_\sigma \\
\uparrow f & & \uparrow \\
f^{-1}(U) & \xrightarrow{\phi_\sigma^f} & X_{\Delta^\sigma}
\end{array} ,
$$

where the horizontal morphisms are Γ_σ-smooth. Set

$$\mathrm{reg}(\sigma) := \langle e_1, \ldots, e_{n-\dim(N_\sigma)} \rangle = \langle e_1, \ldots, e_{\dim(\mathrm{strat}_X(\sigma))} \rangle.$$

Then

1. *there is a fiber square of étale extensions*

$$
\begin{array}{ccc}
U & \xrightarrow{\widetilde{\phi}_\sigma} & X_{\widetilde{\sigma}} \\
\uparrow f & & \uparrow \\
f^{-1}(U) & \xrightarrow{\widetilde{\phi}_\sigma^f} & X_{\widetilde{\Delta}^\sigma},
\end{array}
$$

where the horizontal morphisms are Γ_σ-étale and where

$$\widetilde{\sigma} := \overline{\sigma} \times \mathrm{reg}(\sigma), \quad and \quad \widetilde{\Delta}^\sigma := \Delta^\sigma \times \mathrm{reg}(\sigma).$$

2. *$X_{\widetilde{\sigma}}$ is a Γ_σ-stratified toric variety with the strata described by embedded Γ_σ-semifan $\sigma \subset \widetilde{\sigma}$. Moreover the strata on U are exactly inverse images of strata of $X_{\widetilde{\sigma}}$.*
3. *There is a fiber square of isomorphisms*

$$
\begin{array}{ccc}
\widehat{X}_x & \overset{\widehat{\phi}_\sigma}{\underset{\simeq}{}} & \widetilde{X}_\sigma \\
\uparrow \widehat{f}_x & & \uparrow \\
Y \times_X \widehat{X}_x & \overset{\widehat{\phi}_\sigma^f}{\underset{\simeq}{}} & \widetilde{X}_{\Delta^\sigma}
\end{array}
$$

where

$$\widetilde{X}_\sigma := \widehat{X}_{\widetilde{\sigma}}, \quad and \quad \widetilde{X}_{\Delta^\sigma} := X_{\widetilde{\Delta}^\sigma} \times_{X_{\widetilde{\sigma}}} \widetilde{X}_\sigma.$$

4. *$\widetilde{X}_\sigma$ is a Γ_σ-stratified toroidal scheme with the strata described by embedded Γ_σ-semifan $\sigma \subset \widetilde{\sigma}$. The isomorphism $\widehat{\phi}_\sigma$ preserves strata.*
5. *The morphism $\widehat{f}_x : Y \times_X \widehat{X}_x \to \widehat{X}_x$ is $\mathrm{Aut}^\Gamma(\widehat{X}_x, S)^0$-equivariant .*
6. *The morphism $\widetilde{X}_{\Delta^\sigma} \to \widetilde{X}_\sigma$ is $\mathrm{Aut}(\widetilde{X}_\sigma)^0$-equivariant.* $\square$

Remark. In what follows we assume that n in the definition of $\widetilde{\sigma}$, $\widetilde{\Delta}$, $\widetilde{X}_\sigma$, and $\widetilde{X}_\Delta$ denotes the dimension of relevant stratified toroidal varieties. If the semicomplexes are not associated to stratified toroidal varieties then n will denote some, a priori given, natural number, satisfying the condition that all simplices and semicomplexes have dimension $\leq n$. Thus the definition of $\widetilde{\sigma}$, $\widetilde{\Delta}$, $\widetilde{X}_\sigma$, and $\widetilde{X}_\Delta$ will make sense for any cones σ and any fans Σ considered.

Lemma 4.13.2. *Let Σ be an oriented Γ-semicomplex and $\sigma \in \Sigma$. Let $\phi_1, \phi_2 : (U, S) \to (X_\sigma, S_\sigma)$ be smooth morphisms of Γ-stratified toroidal schemes which determine the same orientation at a closed point $x \in \mathrm{strat}_X(\sigma)$. Let Δ^σ be a subdivision of σ such that $\widetilde{X}_{\Delta^\sigma} \to \widetilde{X}_\sigma$ is $\mathrm{Aut}(\widetilde{X}_\sigma, S_\sigma)^0$-equivariant. Let $\widetilde{U}_i = U \times_{X_\sigma} X_{\Delta^\sigma}$ denote the fiber products for ϕ_i, where $i = 1, 2$. Then $\widetilde{U}_i$ are isomorphic over some neighborhood $V \subset U$ of $x \in X$.*

Proof. Identical with the proof of Lemma 4.12.11.   □

We shall assign to the faces of an oriented Γ-semicomplex Σ the collection of connected proalgebraic groups

$$G^\sigma = \mathrm{Aut}(\widetilde{X}_\sigma)^0.$$

Definition 4.13.3. A proper subdivision $\Delta = \{\Delta^\sigma \mid \sigma \in \Sigma\}$ of an oriented Γ-semicomplex Σ is called *canonical* if for any $\sigma \in \Sigma$, G^σ acts on $\widetilde{X}_{\Delta^\sigma}$ (as an abstract group) and the morphism $\widetilde{X}_{\Delta^\sigma} \to \widetilde{X}_\sigma$ is G^σ-equivariant.

Remark. This technical definition is replaced by a simple combinatorial definition by using the notion of stable support (see section 7.6). Also $\widetilde{X}_\sigma$ and $\widetilde{X}_{\Delta^\sigma}$ are replaced with $\widehat{X}_\sigma$ and $\widehat{X}_{\Delta^\sigma}$ (Proposition 7.6.2).

4.14. Correspondence between canonical subdivisions of semicomplexes and toroidal modifications.

Theorem 4.14.1. *Let (X, S) be an oriented Γ-stratified toroidal variety with the associated oriented Γ-semicomplex Σ. There exists a 1-1 correspondence between the toroidal modifications Y of (X, S) and the canonical subdivisions Δ of Σ.*

1. *If Δ is a canonical subdivision of Σ then the toroidal modification associated to it is defined locally by*

$$
\begin{array}{ccc}
U_\sigma & \longrightarrow & X_\sigma \\
\uparrow f & & \uparrow \\
U_\sigma \times_{X_\sigma} X_{\Delta^\sigma} \quad \simeq \quad f^{-1}(U_\sigma) & \to & X_{\Delta^\sigma}
\end{array}
$$

2. *If $Y^1 \to X$, $Y^2 \to X$ are toroidal modifications associated to canonical subdivisions Δ_1 and Δ_2 of Σ then the natural birational map $Y^1 \to Y^2$ is a morphism iff Δ_1 is a subdivision of Δ_2.*

Proof. We shall construct the canonical subdivision Δ of Σ associated to a given toroidal modification Y.

By Definition 4.12.1 for any point x there is a chart and a subdivision Δ^σ of σ and the fiber square stated in property (1). For two charts $\phi_{\sigma,1} : U_{\sigma,1} \to X_\sigma$ and $\phi_{\sigma,2} : U_{\sigma,2} \to X_\sigma$ we may get two subdivisions Δ_1^σ and Δ_2^σ of σ. These two charts induce by Lemma 4.13.1 isomorphisms $\widetilde{\phi}_{\sigma,i} : \widehat{X}_x \to \widetilde{X}_\sigma$ and their liftings $\widetilde{\phi}_i^f : Y \times_X \widehat{X}_x \simeq \widetilde{X}_{\Delta_1^\sigma} \simeq \widetilde{X}_{\Delta_2^\sigma}$. The isomorphisms $\widetilde{\phi}_{\sigma,i}$ define an automorphism $\alpha := \widetilde{\phi}_{\sigma,2} \widetilde{\phi}_{\sigma,1}^{-1}$ of $\widetilde{X}_\sigma$ preserving orientation and strata, and its lifting isomorphism $\alpha_Y := \widetilde{\phi}_{\sigma,2}^f (\widetilde{\phi}_{\sigma,1}^f)^{-1} : \widetilde{X}_{\Delta_1^\sigma} \to \widetilde{X}_{\Delta_2^\sigma}$. On the other hand α lifts to an automorphism α^f of $\widetilde{X}_{\Delta_1^\sigma} \simeq Y \times_X \widehat{X}_x$. Finally the isomorphism $\alpha_Y' \alpha_Y (\alpha^f)^{-1} : \widetilde{X}_{\Delta_1^\sigma} \to \widetilde{X}_{\Delta_2^\sigma}$ is a lifting of the identity morphism on $\widetilde{X}_\sigma$. Thus it is an identity morphism on $\widetilde{X}_{\Delta_1^\sigma} = \widetilde{X}_{\Delta_2^\sigma}$ and $\Delta_1^\sigma = \Delta_2^\sigma$. This implies the uniqueness of Δ^σ. By Lemma 4.13.1(6), $i_x : \widetilde{X}_{\Delta^\sigma} \to \widetilde{X}_\sigma$ is $\mathrm{Aut}(\widetilde{X}_\sigma)^0$-equivariant. Thus the fan Δ^σ is $\mathrm{Aut}(\sigma)^0$-invariant. It follows from the uniqueness that if $\tau \leq \sigma$ then $\Delta^\sigma|\tau = i_\tau^\sigma(\Delta^\tau) \simeq \Delta^\tau$. Therefore the subdivisions Δ^σ define the canonical subdivision Δ of Σ.

Now let Δ be a canonical subdivision of the oriented Γ-semicomplex Σ. We shall construct the toroidal modification of (X, S) associated to Δ. For any chart $U_{\sigma,a} \to X_\sigma$ set

$$\widetilde{U}_{\sigma,a} := U_{\sigma,a} \times_{X_\sigma} X_{\Delta^\sigma}$$

For any two charts $\phi_{\sigma,a} : U_{\sigma,a} \to X_\sigma$ and $\phi_{\tau,b} : U_{\tau,b} \to X_\tau$ set

$$\widetilde{U}_{\sigma,\tau,a,b} := (U_{\sigma,a} \cap U_{\tau,b}) \times_{X_\sigma} X_{\Delta^\sigma},$$

where the fiber product is defined for the restriction of $\phi_{\sigma,a} : U_{s,a} \to X_\sigma$ to $U_{\sigma,a} \cap U_{\tau,b}$.

It follows from Lemma 4.12.11 that $\widetilde{U}_{\sigma,\tau,a,b}$ is isomorphic over $U_{\sigma,a} \cap U_{\tau,b}$ to $\widetilde{U}_{\tau,\sigma,b,a}$. Thus we can glue all the sets $\widetilde{U}_{\sigma,a}$ along $\widetilde{U}_{\sigma,\tau,a,b}$ and get a separated variety Y. Note that the action of Γ lifts to any subset $\widetilde{U}_{\sigma,a}$ and hence to an action on the whole Y.

A chart $\phi_\sigma : U_\sigma \to X_\sigma$ defines a Γ_σ-equivariant isomorphism $\widehat{X}_x \xrightarrow{\alpha_1} \widetilde{X}_\sigma$. For any $x \in \mathrm{strat}_X(\sigma)$ any automorphism α of $\widehat{X}_x \simeq \widetilde{X}_\sigma$ preserving strata and orientation can be lifted to a Γ_σ-equivariant automorphism α' of $\widehat{X}_y \times_X Y \simeq \widetilde{X}_{\Delta^\sigma}$.

(1) Follows from the construction of a toroidal modification from a canonical subdivision.

(2) Let $Y^1 \to X$, $Y^2 \to X$ be toroidal modifications associated to canonical subdivisions Δ_1 and Δ_2. If Δ_1 is a subdivision of Δ_2 then the natural birational map $Y^1 \to Y^2$ is a morphism since it is a morphism for each chart. On the other hand if $Y^1 \to Y^2$ is a morphism then $Y^1 \times_X \widehat{X}_x \simeq \widetilde{X}_{\Delta_1^\sigma} \to Y^2 \times_X \widehat{X}_x \simeq \widetilde{X}_{\Delta_2^\sigma}$ is a toric morphism. By Lemma 4.12.9 the latter morphism is induced by a toric morphism $X_{\Delta_1^\sigma} \to X_{\Delta_2^\sigma}$. Hence Δ_1^σ is a subdivision of Δ_2^σ for each $\sigma \in \Sigma$ and consequently Δ_1 is a subdivision of Δ_2. This completes the proof of Theorem 4.14.1. $\qquad\square$

5. Stable valuations

For simplicity we consider only valuations with integral values.

5.1. Monomial valuations.

Definition 5.1.1. Let R be a local noetherian ring, $u_1, \ldots, u_k$ be generators of its maximal ideal, and let $a_1, \ldots, a_k$ be nonnegative integers. For any $a \in \mathbf{Z}$ set

$$J_a := (u^{i_1, \cdots, i_k} \mid i_1 a_1 + \ldots + i_k a_k \geq a) \subset R,$$

where $i_1, \ldots, i_k$, is a multiindex in $\mathbf{Z}_{\geq 0}^k$. We call a valuation ν *monomial* with basis $u_1, \ldots, u_k$ and weights $a_1, \ldots, a_k$ if for any $f \in R \setminus \{0\}$,

$$\nu(f) = \max\{a \in \mathbf{Z} \mid f \in J_a\}.$$

Lemma 5.1.2. *Let ν be a nonnegative monomial valuation of a local noetherian ring R. Then for any $a \in \mathbf{Z}$, $I_{\nu,a} = J_a$.* $\qquad\square$

Lemma 5.1.3. *Let ν be a nonnegative valuation of a local noetherian ring R, and $u_1, \ldots, u_k$ be generators of the maximal ideal of R. Let ν_0 be a monomial valuation with basis $u_1, \ldots, u_k$ such that $\nu_0(u_i) = \nu(u_i) \geq 0$. Then $\nu_0(f) \leq \nu(f)$ for any $f \in R$.*

Proof. $I_{\nu_0,a} := \{f \in R \mid \nu_0(f) \geq a\} = J_a \subset I_{\nu,a}$. $\qquad\square$

Lemma 5.1.4. *Let ν be a nonnegative monomial valuation of a local noetherian ring R with basis $u_1, \ldots, u_k$. Let g be an automorphism of R such that $\nu(g(u_i)) \geq \nu(u_i)$ for any $i = 1, \ldots, k$. Then $\nu(g(f)) = \nu(u_i)$ for any $f \in R$.*

Proof. Note that $\nu(g(u^{i_1, \cdots, i_k})) \geq \nu(g(u^{i_1, \cdots, i_k}))$. By Lemma 5.1.2, $g(I_{\nu,a}) = g(J_a) \subset I_{\nu,a}$. If $g(I_{\nu,a}) \subset \neq I_{\nu,a}$ then we obtain an infinite ascending chain of ideals $I_{\nu,a} \subset g^{-1}(I_{\nu,a}) \subset g^{-2}(I_{\nu,a}) \subset \ldots$. $\qquad\square$

5.2. Toric and locally toric valuations.
Let X be an algebraic variety and ν be a valuation of the field $K(X)$ of rational functions. By the valuative criterion of separatedness and properness the valuation ring of ν dominates the local ring of a uniquely determined point (in general nonclosed) c_ν on a complete variety X. (If X is not complete such a point may not exist). We call the closure of c_ν the *center of the valuation* ν and denote it by $Z(\nu)$ or $Z(\nu, X)$. For any $x \in Z(\nu)$ and $a \in \mathbf{Z}_{\geq 0}$,

$$I_{\nu,a,x} := \{f \in \mathcal{O}_{X,x} \mid \nu(f) \geq a\}$$

is an ideal in $\mathcal{O}_{X,x}$. For fixed a these ideals define a coherent sheaf of ideals $\mathcal{I}_{\nu,a}$ supported at $Z(\nu)$.

Lemma 5.2.1. *Let v be an integral vector in the support of the fan Σ. Then the toric valuation $\mathrm{val}(v)$ is centered on $\overline{O}_\sigma$, where σ is the cone whose relative interior contains v.*

Proof. Let $I_{O_\sigma} \subset K[X_\sigma]$ be the ideal of the orbit O_σ. Let $f = \sum a_i x^{m_i}$, $m_i \in \sigma^\vee$, be a regular function on X_σ. Then $\mathrm{val}(v)(f) > 0$ iff $(v, m_i) > 0$ for all $a_i \neq 0$. But $(v, m_i) \geq 0$ by definition and $(v, m_i) = 0$ iff $m_i \in \sigma^\perp$. Thus we have $(v, m_i) > 0$ iff $m_i \in \sigma^\vee \setminus \sigma^\perp$. Finally $I_{O_\sigma} = \{f \in K[X_\sigma] \mid \mathrm{val}(v)(f) > 0\}$ and the valuation ring dominates the local ring of O_σ. $\qquad\square$

Let L be a ring containing K, σ be a cone of the maximal dimension in N_σ and v be an integral vector of σ. Toric valuation val(v) can be defined on any ring $L[[\sigma^\vee]]$ and its arbitrary subring R containing $L[\sigma^\vee]$. For any $f = \sum a_i x^{m_i}$, set

$$\mathrm{val}(v)(f) = \min\{(m_i, v) \mid a_i \neq 0\}.$$

The valuation val(v) of R will be sometimes denoted by val(v, R) or val$(v, \mathrm{Spec}(R))$.

Lemma 5.2.2. *Let σ be a cone of a maximal dimension and T_σ denote the "big torus" acting on $\widehat{X}_\sigma$. Let μ be a T_σ-invariant valuation on $\widehat{X}_\sigma$. Then there exists an integral vector $v \in \sigma$ such that val$(v) = \mu$.*

Proof. The valuation μ defines a linear function on $\sigma^\vee$ corresponding to an integral vector $v \in \sigma$. Since μ is T_σ invariant so are the ideals $I_{\mu,a}$. Therefore the ideals $I_{\mu,a}$ are generated by characters x^m, where $(m, v) \geq a$. Consequently, $I_{\mu,a} = I_{\mathrm{val}(v),a}$ and $\mu = \mathrm{val}(v)$. $\square$

Lemma 5.2.3. *Let σ be a cone of the maximal dimension in a lattice N_σ and $v \in \sigma$ be its integral vector. Let R be a subring of $K[\widehat{X}_\sigma]$ that contains $K[X_\sigma]$. Then for any $a \in \mathbf{Z}$ we have $I_{\mathrm{val}(v,X_\sigma),a} \cdot R = I_{\mathrm{val}(v,R),a}$*

Proof. Denote by R' the localization of R at the maximal ideal of O_σ. Then $I_{\mathrm{val}(v,R'),a} = I_{\mathrm{val}(v,\widehat{X}_\sigma),a} \cap R' = I_{\mathrm{val}(v,X_\sigma),a} \cdot K[\widehat{X}_\sigma] \cap R'$. Since $K[\widehat{X}_\sigma]$ is faithfully flat over R' we obtain $I_{\mathrm{val}(v,X_\sigma),a} \cdot K[\widehat{X}_\sigma] \cap R' = I_{\mathrm{val}(v,X_\sigma),a} \cdot R'$. Thus we have $I_{\mathrm{val}(v,R'),a} = I_{\mathrm{val}(v,X_\sigma),a} \cdot R'$. Then $I_{\mathrm{val}(v,R),a} = I_{\mathrm{val}(v,R'),a} \cap R = I_{\mathrm{val}(v,X_\sigma),a} \cdot R' \cap R = I_{\mathrm{val}(v,X_\sigma),a} \cdot R$. $\square$

Lemma 5.2.4. *Let X_Σ be a toric variety with a toric action of Γ. Let $f : U \to X_\Sigma$ be a smooth Γ-equivariant morphism. Let $v \in \mathrm{int}(\sigma)$, where $\sigma \in \Sigma$, be an integral vector. Assume that the inverse image of $\overline{O}_\sigma$ is irreducible. Then there exists a Γ-invariant valuation μ on U, such that $\mathcal{I}_{\mu,a} = f^{-1}(\mathcal{I}_{\mathrm{val}(v),a}) \cdot \mathcal{O}_U$.*

Proof. Let x belong to the support supp$(f^{-1}(\mathcal{I}_{\mathrm{val}(v),a}) \cdot \mathcal{O}_U) = f^{-1}(\overline{O}_\sigma)$. Then $f(x) \in O_\tau$, where $\tau \geq \sigma$. Let $\pi : X_\tau \to X_{\underline{\tau}}$ denote the natural projection. Then $\mathcal{I}_{\mathrm{val}(v,X_\tau),a} = \pi^{-1}(\mathcal{I}_{\mathrm{val}(v,X_{\underline{\tau}}),a}) \cdot \mathcal{O}_{X_\tau}$. Thus it suffices to prove the lemma replacing X_Σ with $X_{\underline{\tau}}$ and U with the inverse image U' of X_τ. Let $U_x \subset U'$ be an open neighborhood of x for which there exists an étale extension $\widetilde{f} : U_x \to X_{\underline{\tau} \times \mathrm{reg}(\underline{\tau})}$. It defines an isomorphism $\widehat{f} : \widehat{X}_x \simeq \widehat{X}_{\underline{\tau} \times \mathrm{reg}(\underline{\tau})}$. The toric valuation val$(v, X_{\underline{\tau}})$ defines the toric valuation val$(v, X_{\underline{\tau} \times \mathrm{reg}(\underline{\tau})})$ and the corresponding valuation $\overline{\mu}_x$ on $\widehat{X}_x$. By Lemma 5.2.3, $\overline{\mu}_x$ determines the valuation μ_x of $\mathcal{O}_{X,x}$ such that $(\mathcal{I}_{\mu_x,a})_x = f^{-1}(\mathcal{I}_{\mathrm{val}(v),a}) \cdot \mathcal{O}_{X,x}$. Thus μ_x is a valuation on U_x supported on the irreducible set $\overline{f^{-1}(O_\sigma)}$. Note that both sheaves of ideals $\mathcal{I}_{\mu_x,a}$ and $f^{-1}(\mathcal{I}_{\mathrm{val}(v),a}) \cdot \mathcal{O}_X$ are equal in some open neighborhood V_x of x. Therefore the valuation μ_y is the same for all closed point $y \in \overline{f^{-1}(O_\sigma)} \cap V_x$. Since $\overline{f^{-1}(O_\sigma)}$ is irreducibe the valuations μ_x are the same for all $x \in \overline{f^{-1}(O_\sigma)}$ and define a unique valuation μ. Since the sheaves of ideals $f^{-1}(\mathcal{I}_{\mathrm{val}(v),a}) \cdot \mathcal{O}_X = \mathcal{I}_{\mu,a}$ are Γ-invariant μ is also Γ-invariant. $\square$

Let $f : X \to Y$ be a dominant morphism and ν be a valuation of $K(X)$. Then $f_*(\nu)$ denotes the valuation which is the restriction of ν to $K(Y) \simeq f^*(K(Y)) \subset K(X)$.

Let ν be a valuation on X and $\mathcal{I}_{\nu,a}$ be the associated sheaves of ideals. Let $f : Y \to X$ be a morphism for which $f^{-1}(\mathcal{I}_{\nu,a}) \cdot \mathcal{O}_Y$ determine a unique valuation μ on Y such that for sufficiently divisible integer a, $\mathcal{I}_{\mu,a} = f^{-1}(\mathcal{I}_{\nu,a}) \cdot \mathcal{O}_Y$. We denote this valuation by $\mu = f^*(\nu)$.

Definition 5.2.5. By a *locally toric valuation* on an Γ-toroidal variety X we mean a valuation ν of $K(X)$ such that for any point $x \in Z(\nu, X)$ there exists a Γ-equivarinat Γ_x-smooth morphism $\phi : U \to X_\sigma$ from an open Γ-invariant neighborhood U of x to a toric variety X_σ with a toric action of Γ such that $\mu = \phi^*(\mathrm{val}(v))$, where v is an integral vector of σ.

If μ is a locally toric valuation on X then by $\mu_{|\widehat{X}_x}$ we denote the induced valuation on $\widehat{X}_x \simeq \widehat{X}_\sigma$.

Definition 5.2.6. By the *blow-up* bl$_\nu(X)$ of X at a locally toric valuation ν we mean the normalization of

$$\mathrm{Proj}(\mathcal{O} \oplus \mathcal{I}_{\nu,1} \oplus \mathcal{I}_{\nu,2} \oplus \dots),$$

Lemma 5.2.7. *For any natural l, $\mathrm{Proj}(\mathcal{O} \oplus \mathcal{I}_{\nu,1} \oplus \mathcal{I}_{\nu,2} \oplus \dots) = \mathrm{Proj}(\mathcal{O} \oplus \mathcal{I}_{\nu,l} \oplus \mathcal{I}_{\nu,2l} \oplus \dots)$.* $\square$

Denote by bl$_{\mathcal{J}}(X) \to X$ the blow-up of any coherent sheaf of ideals $\mathcal{J}$.

Lemma 5.2.8. *Let X_Σ be a toric variety associated to a fan Σ in N and $v \in |\Sigma| \cap N$. Then $\mathrm{bl}_{\mathrm{val}(v)}(X_\Sigma)$ is the toric variety associated to the subdivision $\langle v \rangle \cdot \Sigma$. Moreover for any sufficiently divisible integer d, $\mathrm{bl}_{\mathrm{val}(v)}(X_\Sigma)$ is the normalization of the blow-up of $\mathcal{I}_{\nu,d}$.*

Proof. It follows from Definition 5.2.6 that $\mathrm{bl}_{\mathrm{val}(v)}(X_\Sigma)$ is a normal toric variety. By Lemma 5.2.1, the sheaf of ideals $\mathcal{I}_{\mathrm{val}(v),d}$ is supported on $\overline{O}_\sigma$, where $v \in \mathrm{int}(\sigma)$, $\sigma \in \Sigma$. By [39], Ch.I, Th. 9, it determines a convex piecewise linear function $f := \mathrm{ord}_{\mathcal{I}_{\mathrm{val}(v),d}} : |\Sigma| \to \mathbf{R}$ such that that $f = 0$ on any $\tau \in \Sigma$ that does not contain v, and $f(u) = \min\{(m, u) \mid m \in \tau^\vee \cap M, (m, v) \geq d\}$ for any τ containing v. Assume that τ contains v and let τ' be a face of τ that does not contain v. Let $m_{\tau',\tau} \in \tau^\vee$ be a nonnegative integral functional on τ which equals 0 exactly on τ'. Assume that d divides all $(m_{\tau',\tau}, v)$. Then $f_{|\tau'+\langle v \rangle} = \frac{d}{(m_{\tau',\tau}, v)} \cdot m_{\tau',\tau}$. Consequently, $\mathcal{I}_{\mathrm{val}(v),d}^k = \mathcal{I}_{\mathrm{val}(v),kd}$ and $\mathrm{bl}_{\mathrm{val}(v)}(X_\Sigma) = \mathrm{bl}_{\mathcal{I}_{\mathrm{val}(v),d}}(X_\Sigma)$. By [39] Ch II Th. 10, the blow-up of $\mathcal{I}_{\mathrm{val}(v),d}$ corresponds to the minimal subdivision Σ' of Σ such that f is linear on each cone in Σ'. By the above,

$$\Sigma' = \Sigma \setminus \mathrm{Star}(\sigma, \Sigma) \cup \bigcup_{\tau' \in \overline{\mathrm{Star}(\sigma,\Sigma)} \setminus \mathrm{Star}(\sigma,\Sigma)} \tau' + \langle v \rangle = \langle v \rangle \cdot \Sigma.$$

$\square$

Proposition 5.2.9. *For any locally toric valuation ν on X there exists an integer d such that*

- *$\mathrm{bl}_\nu(X) = \mathrm{bl}_{\mathcal{I}_{\nu,d}}(X)$.*
- *The valuation ν is induced by an irreducible Q-Cartier divisor on $\mathrm{bl}_\nu(X)$.*

Proof. By quasicompactness of X one can find a finite open affine covering U_i of X of charts $\phi_i : U_i \to X_{\sigma_i}$ such that the valuation ν on each $U_i \subset X$ is equal to $\phi_i^*(\mathrm{val}(v_i))$. It follows from Lemma 5.2.8 that for any i there exists d_i that $\mathrm{bl}_\nu(U_i) = \mathrm{bl}_{\mathcal{I}_{\nu,d_i}}(U_i)$. It suffices to take d equal the product of all d_i. $\square$

5.3. Stable vectors and stable valuations.

Definition 5.3.1. Let (X, S) be an oriented Γ-stratified toroidal variety. A locally toric Γ-invariant valuation ν is *stable* on X if

1. $\mathrm{Z}(\nu, X) = \overline{s}$ for some stratum $s \in S$.
2. For any $x \in \overline{s}$, $\nu_{|\widehat{X}_x}$ is invariant with respect to any Γ_s-equivariant automorphism of $(\widehat{X}_x, S)$, preserving orientation.

Let $f : Y \to (X, S)$ be a toroidal modification of an oriented Γ-stratified toroidal variety. Then a valuation ν on Y is called *X-stable* if $f_*(\nu)$ is stable on (X, S).

Definition 5.3.2. Let Σ be an oriented Γ-semicomplex. A vector $v_0 \in \mathrm{int}(\sigma_0)$, where $\sigma_0 \in \Sigma$, is Σ-*stable* on Σ if for any $\tau \geq \sigma_0$ the corresponding valuation $\mathrm{val}(\imath_{\sigma_0}^\tau(v))$ on $\widetilde{X}_\tau$ is G^τ-invariant. A vector $v \in \sigma$, where $\sigma \in \Sigma$ is Σ-*stable* if there is a stable vector $v_0 \in \mathrm{int}(\sigma_0)$, where $\sigma_0 \leq \sigma$, for which $v = \imath_{\sigma_0}^\sigma(v_0)$.

Let Σ be an oriented Γ-semicomplex. For any $\sigma \in \Sigma$ set

$$\mathrm{stab}(\sigma) := \{a \cdot v \mid v \in \sigma, \ \ v \ \text{ is a stable vector }, a \in \mathbf{Q}_{\geq 0}\}.$$

Lemma 5.3.3.   1. *Stable vectors $v \in \sigma$ are $\mathrm{Aut}(\sigma)^0$-invariant.*
2. *For any $\varrho \geq \tau \geq \sigma$ and any vector $v \in \mathrm{stab}(\sigma)$, $\imath_\tau^\varrho \imath_\sigma^\tau(v) = \imath_\sigma^\varrho(v)$.*
3. *$\imath_\sigma^\tau(\mathrm{stab}(\sigma)) = \imath_\sigma^\tau(\sigma) \cap \mathrm{stab}(\tau)$ for any $\tau \geq \sigma$.*

Proof.(1) Follows from definition.

(2) By definition there is $v_0 \in \mathrm{int}(\sigma_0)$ for which $v = \imath_{\sigma_0}^\sigma(v_0)$. First we prove that for any $\tau \geq \sigma$, we have $\imath_\sigma^\tau \imath_{\sigma_0}^\sigma(v_0) = \imath_{\sigma_0}^\tau(v_0)$.

By Definition 4.11.1 there is an automorphism $\alpha_0 \in \mathrm{Aut}(\sigma_0)^0$ which preserves the orientation such that $\imath_\sigma^\tau \imath_{\sigma_0}^\sigma = \imath_{\sigma_0}^\tau \alpha_0$. By Definition 5.3.2 the valuation $\mathrm{val}(v_0)$ is G^{σ_0}-invariant on $\widetilde{X}_{\sigma_0}$ and hence v_0 is preserved by α_0. This gives $\imath_\sigma^\tau \imath_{\sigma_0}^\sigma(v_0) = \imath_{\sigma_0}^\tau \alpha_0(v_0) = \imath_{\sigma_0}^\tau(v_0)$. Now let $\varrho \geq \tau \geq \sigma$. By the above we have $\imath_\sigma^\varrho \imath_{\sigma_0}^\sigma(v_0) = \imath_{\sigma_0}^\varrho(v_0)$ Then $\imath_\tau^\varrho \imath_\sigma^\tau(v) = \imath_\tau^\varrho \imath_{\sigma_0}^\tau(v_0) = \imath_{\sigma_0}^\varrho(v_0)$.

Finally $\imath_{\sigma_0}^\varrho(v_0) = \imath_\sigma^\varrho \imath_{\sigma_0}^\sigma(v_0) = \imath_\sigma^\varrho(v)$.

(3) If v is a stable vector on σ then by Definition 5.3.2 there is $\sigma_0 \leq \sigma$ and a stable vector $v_0 \in \mathrm{int}(\sigma_0)$ such that $v = \imath^\sigma_{\sigma_0}(v_0)$. Then for $\tau \geq \sigma$, $\imath^\tau_{\sigma_0}(v_0) = \imath^\tau_\sigma \imath^\sigma_{\sigma_0}(v_0) = \imath^\tau_\sigma(v)$ is stable on τ. If $w \in \imath^\tau_\sigma(\sigma) \cap \mathrm{stab}(\tau)$ then there is $\sigma_0 \leq \sigma$ and a stable vector $v_0 \in \mathrm{int}(\sigma_0)$ such that $w = \imath^\tau_{\sigma_0}(v_0) = \imath^\tau_\sigma \imath^\sigma_{\sigma_0}(v_0) = \imath^\tau_\sigma(v)$, where $v = \imath^\sigma_{\sigma_0}(v_0)$. $\qquad\square$

Example 5.3.4. Let Σ be a regular semicomplex and $\sigma = \langle e_1, \ldots, e_k \rangle \in \Sigma$. Then $e := e_1 + \ldots + e_k$ is stable since it corresponds to the valuation of the stratum $\mathrm{strat}_X(\sigma)$ on $\widetilde{X}_\sigma$. The subset s defines a smooth subvariety in a smooth neighborhood and therefore it determines a valuation. If we blow-up the subvariety $\mathrm{strat}_X(\sigma)$ then this valuation coincides with the one defined by the exceptional divisor.

Example 5.3.5. Consider the stratified toroidal variety $\mathbf{A}^2$ from Example 3.1.13. The valuation of the point $(0,0)$ corresponds to the vector $(1,1) \in N$, the valuation of $\overline{s} = \mathbf{A}^1 \times \{0\}$ corresponds to the vector $(0,1) \in N$. Then $|\mathrm{Stab}(\Sigma)| = \langle (1,1), (0,1) \rangle$. The valuations from $\langle (1,0), (0,1) \rangle \setminus \langle (1,1), (0,1) \rangle$ are not stable: Let $\widetilde{\mathbf{A}^2} \to \mathbf{A}^2$ be the blow-up of the point $(0,0) \in \mathbf{A}^2$. The valuations from $\langle (1,0), (0,1) \rangle \setminus \langle (1,1), (0,1) \rangle$ are centered on $\widetilde{\mathbf{A}^2}$ at $l_2 \cap D$ or l_2 (using notation from Example 3.1.13) and the automorphism $\phi : \mathbf{A}^2 \to \mathbf{A}^2$, $\phi(x_1, x_2) = (x_1 + x_2, x_2)$, preserves strata and moves the subvarieties l_2 and $l_2 \cap D$. In particular it changes the relevant valuations.

Lemma 5.3.6. *Let (X, S) be a Γ-stratified toroidal variety with the associated oriented Γ-semicomplex Σ. The following conditions are equivalent:*

1. *ν is stable on X.*
2. *There exists a stable vector $v \in \mathrm{int}(\sigma)$, where $\sigma \in \Sigma$, such that for any $\tau \geq \sigma$ and any chart $\phi : U \to X_\tau$ we have $\nu = \phi^*(\mathrm{val}(\imath^\tau_\sigma(v)))$.*

Proof. Let x denote a closed point of $\overline{\mathrm{strat}_X(\sigma)}$. Then $x \in \mathrm{strat}_X(\tau)$, where $\tau \geq \sigma$. The valuation ν on X determines a G^τ-invariant valuation on $\widehat{X}_x \simeq \widetilde{X}_\tau$ which is in particular a toric valuation so, by Lemmas 5.2.2 and 5.2.1 it corresponds to a unique $v \in \mathrm{int}(\sigma)$. Thus v is stable and ,by Lemma 5.2.4, $\nu = \phi^*(\mathrm{val}(v))$.

Let v be a stable vector. The valuation $\mathrm{val}(v)$ determines a monomial valuation $\nu := \widehat{\phi}^*(\mathrm{val}(v))$ on $\widehat{X}_x$, where $x \in \overline{\mathrm{strat}_X(\sigma)} \cap U$. Since $\mathrm{val}(v)$ is G^τ-invariant on $\widetilde{X}_\tau$ for any $\tau \geq \sigma$ we see that $\nu_{|\widehat{X}_x}$ is invariant with respect to any Γ_x-equivariant automorphism preserving strata and orientation. Moreover $\nu_{|\widehat{X}_x}$ and ν do not depend on ϕ. $\qquad\square$

Lemma 5.3.7. *Let $f : Y \to X$ be a toroidal modification of a Γ-stratified toroidal variety (X, S) associated to a canonical subdivision Δ of Σ. The following conditions are equivalent:*

1. *ν is X-stable on Y.*
2. *There exists a Σ-stable vector $v \in \mathrm{int}(\sigma)$, where $\sigma \in \Sigma$, such that for any $\tau \geq \sigma$ and morphism $\phi^f_\tau : f^{-1}(U_\tau) \to X_{\Delta^\tau}$, induced by a chart $\phi_\tau : U_\tau \to X_\tau$ we have $\nu = \phi^{f*}_\tau(\mathrm{val}(v))$.*

Proof. Let $y \in Z(\nu, Y)$, $x = f(y) \in X$ and $\phi : U \to X_\sigma$ be a chart at x. We can extend this Γ^σ-smooth morphism to an étale morphism $\widetilde{\phi} : U \to X_{\widetilde{\sigma}}$. By Lemma 5.3.6 we have $\nu = \widetilde{\phi}^*(\mathrm{val}(v))$ for some stable vector $v \in \sigma$.

Consider the fiber squares

$$
\begin{array}{ccc}
f^{-1}(U) & \xrightarrow{\widetilde{\phi}_f} & X_{\widetilde{\Delta}} \\
\downarrow f & & \downarrow i \\
U & \xrightarrow{\widetilde{\phi}} & X_{\widetilde{\sigma}}
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y \times \widehat{X}_x & \xrightarrow{\widehat{\phi}_f} & \widetilde{X}_\Delta \\
\downarrow \widehat{f} & & \downarrow \widehat{i} \\
\widehat{X}_x & \xrightarrow{\widehat{\phi}} & \widetilde{X}_\sigma,
\end{array}
$$

where $\widetilde{\phi}^f_\sigma$ is the induced étale morphism and i is a toric morphism. Then $\widehat{\phi}$ as well as $\widehat{\phi^f_\sigma}$ are isomorphisms and we have $\nu = \widehat{f}_*^{-1} \widehat{\phi}^* i_*(\mathrm{val}(v)) = \widehat{\phi}^*_f(\mathrm{val}(v))$ on $Y \times_X \widehat{X}_x$. Consequently $\nu = f_*^{-1} \widetilde{\phi}^* i_*(\mathrm{val}(v)) = \widetilde{\phi}^*_f(\mathrm{val}(v))$ since the above valuations coincide on $f^{-1}(U)$. $\qquad\square$

Lemma 5.3.8. *Let σ and τ be Γ-semicones in isomorphic lattices $N_\sigma \simeq N_\tau$, and L be a field containing K. Consider the induced action of Γ on $L[[\sigma^\vee]]$ and $L[[\tau^\vee]]$ which is trivial on L. Let $\psi : L[[\sigma^\vee]] \simeq L[[\tau^\vee]]$ be a Γ-equivariant isomorphism over K. Let $v_\sigma \in \mathrm{int}(\sigma)$ be a minimal internal vector of σ. Then there exists a minimal internal vector $v_\tau \in \mathrm{int}(\tau)$ such that $\psi_*(\mathrm{val}(v_\sigma)) = \mathrm{val}(v_\tau)$.*

Proof. Note that $\mathrm{val}(v_\sigma)(f_\sigma) > 0$ and $\psi_*(\mathrm{val}(v_\sigma))(f_\tau) > 0$, for any functions $f_\sigma \in m_\sigma$, $f_\tau \in m_\tau$ from the maximal ideals $m_\sigma \subset L[[\sigma^\vee]]$ and $m_\tau \subset L[[\tau^\vee]]$ respectively. The dual cone $\tau^\vee$ of regular characters defines a subgroup in the multiplicative group of rational functions. The valuation $\psi_*(\mathrm{val}(v_\sigma))$ determines a group homomorphism of that subgroup into $\mathbf{Z}$ and hence defines a linear function on $\tau^\vee$ corresponding to a vector $v_\tau \in \mathrm{int}(\tau)$. We have to show that $\mathrm{val}(v_\tau) = \psi_*(\mathrm{val}(v_\sigma))$ (it is not clear whether $\psi_*(\mathrm{val}(v_\sigma))$ is monomial with respect to characters). By Lemma 5.1.3, $\psi_*(\mathrm{val}(v_\sigma)) \geq \mathrm{val}(v_\tau)$. Let $v'_\sigma \in \mathrm{int}(\sigma)$ correspond to the linear function determined by $\psi^{-1*}(\mathrm{val}(v_\tau))$ on the cone $\sigma^\vee$. Then $\mathrm{val}(v_\sigma) = \psi^{-1*}\psi_*(\mathrm{val}(v_\sigma)) \geq \psi^{-1*}(\mathrm{val}(v_\tau)) \geq \mathrm{val}(v'_\sigma)$. Finally, $\mathrm{val}(v_\sigma) \geq \mathrm{val}(v'_\sigma)$. By the minimality of v_σ it follows that $v_\sigma = v'_\sigma$ and $\mathrm{val}(v_\sigma) = \psi^{-1*}(\mathrm{val}(v_\tau)) = \mathrm{val}(v'_\sigma)$, or equivalently $\psi^*(\mathrm{val}(v_\sigma)) = \mathrm{val}(v_\tau)$.

We need to show that v_τ is a minimal internal vector. Suppose it isn't. Write $v_\tau = v_{\tau 0} + v_{\tau 1}$, where $v_{\tau 0} \in \mathrm{int}(\tau)$ and $v_{\tau 1} \in \tau$. By the above find $v_{\sigma 0} \in \mathrm{int}(\sigma)$ such that $\mathrm{val}(v_{\sigma 0}) = \psi^{-1*}(\mathrm{val}(v_{\tau 0}))$. Then $\mathrm{val}(v_\sigma) = \psi^{-1*}(\mathrm{val}(v_\tau)) \geq \psi^{-1*}(\mathrm{val}(v_{\tau 0})) = \mathrm{val}(v_{\sigma 0})$. By the minimality of v_σ, we get $v_\sigma = v_{\sigma 0}$ and $v_\tau = v_{\tau 0}$. $\qquad\square$

Lemma 5.3.9. *Let σ be a Γ-semicone and $v \in \sigma$ be a vector such that for any $\phi \in G^\sigma$ there exists $v_\phi \in \sigma$ such that $\phi^*(\mathrm{val}(v)) = \mathrm{val}(v_\phi)$. Then $\mathrm{val}(v)$ is G^σ-invariant.*

Proof. Set $X := \widetilde{X}_\sigma$, $S := S_{\widetilde{\sigma}}$, $x := O_{\widetilde{\sigma}}$. Let W denote the set of all vectors $v_\phi \in \sigma$ for all $\phi \in \mathrm{Aut}(X, S)$. For any natural a, the ideals $I_{\mathrm{val}(v_\phi),a} := \{f \in \widehat{\mathcal{O}}_{X,x} \mid \mathrm{val}(v_\phi)(f) \geq a\}$ are generated by monomials. They define the same Hilbert-Samuel function $k \mapsto \dim_K(K[X]/(I + m^k))$, where m denotes the maximal ideal. It follows that the set W is finite.

On the other hand since the ideals $I_{\mathrm{val}(v_\phi),a}$ are generated by monomials they can be distinguished by the ideals $\mathrm{gr}(I_{\mathrm{val}(v_\phi),a})$ in the graded ring

$$\mathrm{gr}(\widehat{\mathcal{O}}_{X,x_\sigma}) = \widehat{\mathcal{O}}_{X,x}/m_{x,X} \oplus m_{x,X_\sigma}/m^2_{x,X_\sigma} \oplus \dots,$$

where $m_{x,X} \subset \widehat{\mathcal{O}}_{X,x}$ is the maximal ideal of the point x.

Let $d : \mathrm{Aut}(\widetilde{X}_\sigma, S_\sigma)^0 \to \mathrm{Gl}(\mathrm{Tan}_{X,x})$ be the differential morphism (see Example 4.5.4). Then the connected algebraic group $d(\mathrm{Aut}(\widetilde{X}_\sigma, S_\sigma)^0)$ acts algebraically on the connected component of the Hilbert scheme of graded ideals with fixed Hilbert polynomial (see Example 4.5.4) . In particular it acts trivially on the finite subset $\mathrm{gr}(I_{\mathrm{val}(v_\phi),a})$, and consequently $\mathrm{Aut}(\widetilde{X}_\sigma, S_\sigma)^0$ preserves all $I_{\mathrm{val}(v_\phi),a}$. $\qquad\square$

Lemma 5.3.10. *Let σ be a Γ-semicone and $\widetilde{X}_{\Delta^\sigma} \to \widetilde{X}_\sigma$ be a G^σ-equivariant morphism. Then all its exceptional divisors are G^σ-invariant.*

Proof. Any automorphism $g \in G^\sigma$ maps an exceptional divisor D to another exceptional divisor D'. Hence the valuation val_D defined by D satisfies the condition of the previous lemma, and therefore it is G^σ-invariant. $\qquad\square$

Definition 5.3.11. Let σ be a Γ-semicone. A valuation $\mathrm{val}(v)$, where $v \in \sigma$, is *G^σ-semiinvariant* if for any $\phi \in G^\sigma$ there exist $v' \in \sigma$ such that $\phi^*(\mathrm{val}(v)) = \mathrm{val}(v')$.

By abuse of terminology a vector $v \in \sigma$ will be called *G^σ-semiinvariant* (resp. *G^σ-invariant*) if the corresponding valuation $\mathrm{val}(v)$ is G^σ-semiinvariant (resp. G^σ-invariant).

Definition 5.3.12. Let Σ be an oriented Γ-semicomplex. $v \in \mathrm{int}(\sigma)$ is *semistable* if for any $\tau \geq \sigma$, $\mathrm{val}(v)$ is G^τ-semiinvariant on $\widetilde{X}_\tau$.

Let $\mathrm{Vert}(\Sigma)$ denote the set of all 1-dimensional rays in the fan Σ.

Definition 5.3.13. Let σ be a Γ-semicone and $\widetilde{X}_{\Delta^\sigma} \to \widetilde{X}_\sigma$ be a G^σ-equivariant morphism. A cone $\delta \in \Delta^\sigma$ is *G^σ-invariant* if

1. there is $\tau \leq \sigma$ such that $\mathrm{int}(\delta) \subset \mathrm{int}(\tau)$,
2. $\overline{O}_\delta \subset \widetilde{X}_{\Delta^\sigma}$ is G^σ-invariant.

Definition 5.3.14. Let Δ be a canonical subdivision of an oriented Γ-semicomplex Σ. A cone $\delta \in \Delta^\sigma$, $\sigma \in \Sigma$ is called a *Σ-stable face* of Δ if

1. $\mathrm{int}(\delta) \subset \mathrm{int}(\sigma)$,
2. $\imath^\tau_\sigma(\delta) \in \Delta^\tau$ is G^τ-invariant for any $\tau \geq \sigma$.

Lemma 5.3.15. *Let Let σ be a Γ-semicone and $\widetilde{X}_{\Delta^\sigma} \to \widetilde{X}_\sigma$ be a G^σ-equivariant morphism.*

1. *All G^σ-semiinvariant vectors $v \in \sigma$ are G^σ-invariant.*
2. $\mathrm{Vert}(\Delta^\sigma) \setminus \mathrm{Vert}(\overline{\sigma})$ *are G^σ-invariant.*
3. *Let δ be a G^σ-invariant face of Δ^σ. Then any minimal internal vector $v \in \mathrm{int}(\delta)$ are G^σ-invariant.*
4. *Let δ be a $\underline{\Gamma\text{-}indecomposable}$ face of Δ^σ. Then δ is G^σ-invariant and all its minimal internal vectors v are G^σ-invariant. In particular all minimal generators of any face $\delta \in \Delta^\sigma$ are G^σ-invariant.*
5. *Let $\delta \in \Delta^\sigma$ be a minimal face such that $\mathrm{int}(\delta) \subset \mathrm{int}(\tau)$, where $\tau \neq \{0\}$. Then δ is G^σ-invariant and $\mathrm{int}(\delta)$ contains a G^σ-invariant vector.*

Proof. (1) Follows from Lemma 5.3.9. (2) Follows from Lemma 5.3.10. (3) By Lemma 5.3.8, v is G^σ-semiinvariant. By (1), v is G^σ-invariant.

(4) Let $\tau \preceq \sigma$ be a face of σ such that $\mathrm{int}(\delta) \subset \mathrm{int}(\tau)$. By Lemmas 4.12.8 and 4.12.5, $\Delta^\sigma|\tau = \Delta^\sigma|\omega(\tau) \oplus r(\tau)$ for some regular cone $r(\tau)$. Since δ is Γ-indecomposable we conclude that $\delta \in \Delta^\sigma|\omega(\tau)$ and $\tau = \omega(\tau) \leq \sigma$. By definition G^σ acts on $\widetilde{X}_{\Delta^\sigma}$ and consequently on strata from the stratification $\mathrm{Sing}^\Gamma(\widetilde{X}_{\Delta^\sigma})$ (see Lemma 4.7.4).

Let $\mathrm{sing}_0 \in \mathrm{Sing}(\widetilde{X}_{\Delta^\sigma})$ denote the stratum corresponding to indecomposable face $\delta_0 \in \Delta^\sigma$.

The images of the G^σ action on sing_0 form a finite invariant subset of strata $\mathrm{Sing}_0 \subset \mathrm{Sing}(\widetilde{X}_{\Delta^\sigma})$. All strata from Sing_0 correspond to some isomorphic cones $\delta_i \in \Delta^\sigma$ for $i = 0, \dots, l$. Let $v \in \mathrm{int}(\delta)$ be a minimal internal vector. Then by Lemma 5.3.8 for any $\widetilde{\phi} \in G^\sigma$ the image $\widetilde{\phi}_*(\mathrm{val}(v))$ is a valuation on $\widetilde{X}_{\Delta^\tau}$ corresponding to a minimal internal vector of δ_i. Hence v is G^σ-semiinvariant and by (1) it is G^σ-invariant. By Lemma 2.1.3 each minimal generator is a minimal internal vector of some indecomposable cone δ, hence by the above it is G^σ-invariant.

(5) For any $\tau \geq \sigma$ the cone δ corresponds to the maximal component O_δ in $\widetilde{X}_{\Delta^\sigma}$ which dominates $O_\tau \subset \widetilde{X}_\sigma$. Then G^σ permutes the set of maximal components of the inverse image of O_τ dominating O_τ and we repeat the reasoning from (4). □

Remark. The above lemma holds if we replace G^σ with its connected proalgebraic subgroup.

As a direct corollary of Lemma 5.3.15 is the following

Lemma 5.3.16. *Let Δ be a canonical subdivision of an oriented Γ-semicomplex Σ.*

1. *All the semistable vectors in Σ are stable.*
2. *For any $\sigma \in \Sigma$ all elements of $\mathrm{Vert}(\Delta^\sigma) \setminus \mathrm{Vert}(\overline{\sigma})$ are Σ-stable.*
3. *Let δ be a Σ-stable face of Δ. Then any minimal internal vector $v \in \mathrm{int}(\delta)$ is Σ-stable.*
4. *Let δ be a $\underline{\Gamma\text{-}indecomposable}$ face of Δ^σ. Then δ is Σ-stable and all its minimal internal vectors are stable. In particular all minimal generators of any face $\delta \in \Delta^\sigma$ are Σ-stable.*
5. *Let $\delta \in \Delta^\sigma$ be a minimal face such that $\mathrm{int}(\delta) \subset \mathrm{int}(\sigma)$, where $\sigma \neq \{0\}$. Then δ is Σ-stable and $\mathrm{int}(\delta)$ contains a Σ-stable vector.* □

5.4. Star subdivisions and blow-ups of stable valuations.

Proposition 5.4.1.   1. *Let Σ be an oriented Γ-semicomplex and $v_1, \dots, v_k$ be Σ-stable vectors. Then $\Delta := \langle v_k \rangle \cdot \dots \cdot \langle v_1 \rangle \cdot \Sigma$ is a canonical subdivision of Σ.*

2. *Let (X, S) be a Γ-stratified toroidal variety with an associated oriented Γ-semicomplex Σ. Let $v_1, \dots, v_k$ be stable vectors and $\nu_1, \dots, \nu_k$ be the associated stable valuations. Then the composite of blow-ups $\mathrm{bl}_{\nu_k} \circ \dots \circ \mathrm{bl}_{\nu_1} : X_k \to X$ is the toroidal modification of X associated to the canonical subdivision $\Delta := \langle v_k \rangle \cdot \dots \cdot \langle v_1 \rangle \cdot \Sigma$ of Σ.*

Proof. Induction on k. Let $\Delta_k := \langle v_k \rangle \cdot \dots \cdot \langle v_1 \rangle \cdot \Sigma$ be a canonical subdivision corresponding to $f := \mathrm{bl}_{\nu_k} \circ \dots \circ \mathrm{bl}_{\nu_1} : X_k \to X$. Then $v_{k+1} \in \mathrm{int}(\sigma)$ defines an X-stable valuation ν_{k+1} on X_k. It follows from 5.3.7 that for any chart $\phi_\sigma : U_\sigma \to X_\sigma$ the blow-up of $f^{-1}(U_\sigma) \subset X_j$ at $\nu_{k+1} = \phi^{f*}(\mathrm{val}(v_{k+1}))$ corresponds to the star subdivision of Δ_k^σ at v_{k+1}. □

6. Correspondence between toroidal morphisms and canonical subdivisions

6.1. Definition of stable support.

By Lemma 5.3.3 we are in a position to glue pieces $\mathrm{stab}(\sigma)$ into one topological space.

Definition 6.1.1. The *stable support* of a Γ-semicomplex Σ is the topological space $\mathrm{Stab}(\Sigma) := \bigcup_{\sigma \in \Sigma} \mathrm{stab}(\sigma)$.

Let $G \subset \mathrm{Aut}(\widetilde{X}_\sigma)$ be any abstract algebraic group. Let I_G denote the set of G-invariant toric valuations on $\widetilde{X}_\sigma$. Set

$$\mathrm{Inv}(G, \sigma) := \{a \cdot v \mid \mathrm{val}(v) \in I_G, \ a \in \mathbf{Q}_{\geq 0}\}, \quad \mathrm{Inv}(\sigma) := \mathrm{Inv}(G^\sigma, \sigma).$$

Lemma 6.1.2. *Let Σ be a Γ-semicomplex.*

1. $\mathrm{stab}(\sigma) \cap \mathrm{int}(\sigma) = \bigcap_{\tau \geq \sigma} \mathrm{Inv}(\tau) \cap \mathrm{int}(\sigma)$.
2. $\mathrm{stab}(\sigma) = \bigcup_{\tau \leq \sigma} (\mathrm{stab}(\overline{\tau}) \cap \mathrm{int}(\tau))$.

Proof. Follows from the definitions of stable valuation and stable support. $\square$

6.2. Convexity of the stable support.

Lemma 6.2.1.   1. *For any abstract subgroup $G \subset \mathrm{Aut}(\widehat{X}_\sigma)$ the cone $\mathrm{Inv}(G, \sigma)$ is convex.*
2. *Let Σ be a Γ-semicomplex. Then for any $\sigma \in \Sigma$, $\mathrm{stab}(\sigma)$ is convex.*

Proof. (1). Let $v_1, v_2 \in I_G$ and $\mathrm{bl}_{\mathrm{val}(v_1)} \circ \mathrm{bl}_{\mathrm{val}(v_2)} : X_{\langle v_1 \rangle \cdot \langle v_2 \rangle \cdot \sigma} \to X_\sigma$ be the toric morphism. The induced morphism $(\mathrm{bl}_{\mathrm{val}(v_1)} \circ \mathrm{bl}_{\mathrm{val}(v_2)})\widehat{\ } : X_{\langle v_1 \rangle \cdot \langle v_2 \rangle \cdot \sigma} \times_{X_\sigma} \widehat{X}_\sigma \to \widehat{X}_\sigma$ is G-equivariant. The exceptional divisors D_1, D_2 of $\mathrm{bl}_{\mathrm{val}(v_1)} \circ \mathrm{bl}_{\mathrm{val}(v_2)}$ correspond to $v_1, v_2 \in \sigma$. The D_1 and D_2 intersect along a stratum O_τ corresponding to the cone $\tau := \langle v_1, v_2 \rangle$. Then $\widehat{X}_\tau$ is a G-invariant local scheme of toric variety at the generic point of the orbit O_τ and $\{D_1, D_2, O_\tau\}$ is the orbit stratification on $(\widehat{X}_\tau)$ which is also G-invariant.

Let $u_1, \ldots, u_k$ denote semiinvariant generators of $\widehat{\mathcal{O}}_{X_\tau, O_\tau} \simeq K(O_\tau)[[u_1, \ldots, u_k]]$ Each automorphism from G preserves the orbit stratification and multiplies the generating monomials u_i by invertible functions. Therefore it does not change the valuations $\mathrm{val}(v)$, where $v \in \tau$, on $\widehat{\mathcal{O}}_{X_\tau, O_\tau} \simeq K(O_\tau)[[u_1, \ldots, u_k]]$.

(2) Follows from (1) and from Lemma 6.1.2 $\square$

6.3. Toroidal embeddings and stable support.

Lemma 6.3.1. *Let (X, S) be a toroidal embedding. Then*

1. *All integral vectors in the faces σ of Σ are stable.*
2. *All subdivisions of Σ are canonical.*

Proof. (1) Each automorphism g from G^σ preserves divisors, hence multiplies the generating monomials by invertible functions. Consequently, it does not change the valuations $\mathrm{val}(v)$, where $v \in \sigma$, on $\widetilde{X}_\sigma$.

(2) Let Δ^σ be a subdivision of σ. For any $\delta \in \Delta^\sigma$ each automorphism g of $\widetilde{X}_\sigma$ lifts to an automorphism g' of $\widetilde{X}_\delta = \widetilde{X}_\sigma \times_{X_{\widehat{\sigma}}} X_{\widetilde{\delta}}$ which also multiplies monomials by suitable invertible functions. Therefore g lifts to the scheme $\widetilde{X}_{\Delta^\sigma}$. $\square$

6.4. Minimal vectors and stable support.

Lemma 6.4.1. *Let σ be a Γ-semicone and $\widetilde{X}_{\Delta^\sigma} \to \widetilde{X}_\sigma$ be a G^σ-equivariant morphism.*

1. *Let $\delta \in \Delta^\sigma$ be a simplicial cone where $\sigma \in \Sigma$. Then all vectors in $\mathrm{par}(\delta)$ are Σ-stable.*
2. *Let $\delta \in \Delta^\sigma$ be a G^σ-invariant simplicial face. Then all vectors from $\overline{\mathrm{par}(\delta)} \cap \mathrm{int}(\delta)$ are G^σ-invariant.*

Proof. (1) Each $v \in \mathrm{par}(\delta)$ is a nonnegative integral combination of minimal generators of δ. Minimal generators of δ are G^σ-invariant by Lemma 5.3.15(4). Their linear combination is G^σ-invariant by Lemma 6.2.1.

(2) Let $v \in \overline{\mathrm{par}(\delta)} \cap \mathrm{int}(\delta)$. Write $\delta = \langle w_1, \ldots, w_k \rangle$ and $v = \sum \alpha_i w_i$, where $0 \leq \alpha_i \leq 1$. If v is a minimal internal vector then it is G^σ-invariant by Lemma 5.3.15(3). If not then $v = v' + v''$, where $v' = \sum \beta_i w_i$,

$0 < \beta_i \leq \alpha_i \leq 1$, is a minimal internal vector and $v'' = \sum(\alpha_i - \beta_i)w_i \in \mathrm{par}(\delta)$. By (1) and Lemma 6.2.1, v' is G^σ-invariant. $\qquad\square$

A direct corollary from the abo Lemma is the following Lemma.

Lemma 6.4.2. *Let Δ be a canonical subdivision of an oriented Γ-semicomplex Σ.*

1. *Let $\delta \in \Delta^\sigma$ be a simplicial cone where $\sigma \in \Sigma$. Then all vectors in $\underline{\mathrm{par}}(\delta)$ are Σ-stable.*
2. *Let $\delta \in \Delta^\sigma$ be a Σ-stable simplicial cone. Then all vectors from $\overline{\mathrm{par}(\delta)} \cap \mathrm{int}(\delta)$ are Σ-stable.* $\qquad\square$

6.5. Canonical stratification on a toroidal modification.

Lemma 6.5.1. *Let Δ be a canonical subdivision of an oriented Γ-semicomplex Σ. For any $\sigma \in \Sigma$ set*

$$\Delta^\sigma_{\mathrm{stab}} := \{\delta \in \Delta^\sigma \mid \mathrm{int}(\delta) \cap \mathrm{stab}(\sigma) \neq \emptyset\}.$$

Then

1. *$\Delta^\sigma_{\mathrm{stab}} \subset \Delta^\sigma$ is an embedded Γ_σ-semifan in N_σ.*
2. *The subset $\Delta^\sigma_{\mathrm{stab}} \subset \Delta^\sigma$ consists of all Σ-stable faces in Δ^σ.*
3. *For any $\tau \leq \sigma$, $\Delta^\sigma_{\mathrm{stab}} | \imath^\sigma_\tau(\tau) = \imath^\sigma_\tau(\Delta^\tau_{\mathrm{stab}})$.*
4. *For any $\sigma \in \Sigma$, the stratification S^{stab}_σ on $\widetilde{X}_{\Delta^\sigma}$ determined by the embedded Γ-semifan $\Delta^\sigma_{\mathrm{stab}} \subset \Delta^\sigma$ is G^σ-invariant.*

Proof. (1) By Lemma 6.2.1, $\mathrm{stab}(\sigma)$ is convex and there is a unique maximal face $\omega \preceq \delta$ whose relative interior $\mathrm{int}(\omega)$ intersects the stable support. Thus $\omega \in \Delta^\sigma_{\mathrm{stab}}$. By Lemma 5.3.16(4), either $\mathrm{sing}^\Gamma(\delta) = \{0\}$ or $\mathrm{int}(\mathrm{sing}^\Gamma(\delta)) \cap \mathrm{stab}(\sigma) \neq \emptyset$. In both cases $\mathrm{sing}^\Gamma(\delta) \leq \omega$ and consequently $\delta = \omega \oplus^\Gamma r(\delta)$, for some regular cone $r(\delta)$.

(2) If $\delta \in \Delta^\sigma$ is Σ-stable then by Lemma 5.3.16(3) its minimal internal vector is Σ-stable. Therefore $\delta \in \Delta^\sigma_{\mathrm{stab}}$. If the relative interior of $\delta \in \Delta^\sigma_{\mathrm{stab}}$ contais a stable vector $v \in \mathrm{int}(\tau)$, where $\tau \leq \sigma$, then $\mathrm{int}(\delta) \subset \mathrm{int}(\tau)$, $\delta \in \Delta^\tau_{\mathrm{stab}}$, and for any $\varrho \geq \tau$, the closure of $O_{\imath^\varrho_\tau(\delta)} \subset \widetilde{X}_{\Delta^\varrho}$ is exactly the center of $\mathrm{val}(v)$, and therefore is G^ϱ-invariant.

(3) (4) follow from (2).

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

As a simple corollary of the above we get

Lemma 6.5.2. *There is a Γ-semicomplex Δ_{stab} obtained by glueing the semicomplexes $\Delta^{\sigma\ semic}_{\mathrm{stab}}$ along $\Delta^{\tau\ semic}_{\mathrm{stab}}$, where $\tau \leq \sigma$.* $\qquad\square$

Proof For any $\omega \in \Delta^{\sigma\ semic}_{\mathrm{stab}}$ denote by $\sigma(\omega \leq \sigma$ the semicone in Σ for which $\mathrm{int}(\omega) \subset \mathrm{int}(\sigma(\omega))$. Then $\omega \in \Delta^{\sigma(\omega)\ semic}_{\mathrm{stab}}$. For $\omega \in \Delta^{\sigma(\omega)\ semic}_{\mathrm{stab}}$ and $\gamma \in \Delta^{\sigma(\gamma)\ semic}_{\mathrm{stab}}$, write $\omega \leq \gamma$ if $\sigma(\omega) \leq \sigma(\gamma)$ and $\imath^{\sigma(\gamma)}_{\sigma(\omega)}(\omega) \leq \gamma$. Then for $\omega \leq \gamma$ we set $\imath^\gamma_\omega := \imath^{\sigma(\gamma)}_{\sigma(\omega)}$ and $\Gamma_\omega := (\Gamma_{\sigma(\omega)})_\omega$. $\qquad\square$

Proposition 6.5.3. *Let (X, S) be an oriented Γ-stratified toroidal variety with an associated oriented Γ-semicomplex Σ. Let Y be a toroidal modification of (X, S) corresponding to a canonical subdivision Δ of Σ. Then there is a canonical stratification R on Y with the following properties:*

1. *Let $\phi^f_\sigma : f^{-1}(U_\sigma) \to X_{\Delta^\sigma}$ denote the morphism induced by a chart $\phi_\sigma : U_\sigma \to X_\sigma$. The intersections $r \cap f^{-1}(U_\sigma)$, $r \in R$ are precisely the inverse images of the strata associated to the embedded Γ-semifan $\Delta^\sigma_{\mathrm{stab}} \subset \Delta^\sigma$.*
2. *The closures $\overline{r}$ of strata $r \in R$ are centers of X-stable valuations on Y.*
3. *The Γ-semicomplex associated to (Y, R) is equal to $\Sigma_R = \Delta_{\mathrm{stab}}$ and the atlas is given by $\bigcup_{\sigma \in \Sigma} \mathcal{U}(\Delta^\sigma, \Delta^\sigma_{\mathrm{stab}})$.*
4. *The stratification R is the finest stratification on Y satisfying the conditions:*
   (a) *The morphism f maps strata in R onto strata in S.*
   (b) *There is an embedded semifan $\Omega^\sigma \subset \Delta^\sigma$ such that the intersections $r \cap f^{-1}(U_\sigma)$, $r \in R$, are inverse images of strata of $S_{\Omega^\sigma})$ on $(X_{\Delta^\sigma}$.*
   (c) *For any point x every Γ_x-equivariant automorphism α of $\widehat{X}_x$ preserving strata and orientation can be lifted to an automorphism α' of $Y \times_X \widehat{X}_x \to Y$ preserving strata.*

Proof. For any face $\omega \in \Delta^\sigma_{\mathrm{stab}}$ we find a Σ-stable vector v_ω in its relative interior. By Lemma 5.3.6 the vector v_ω corresponds to an X-stable valuation ν_ω on Y. We define the closure of a stratum $r \in R$ asociated

to ω to be

$$\overline{r} := \overline{\mathrm{strat}_Y(\omega)} := \mathrm{Z}(\nu_\omega).$$

Then define the stratum r as

$$r := \mathrm{strat}_Y(\omega) := \overline{\mathrm{strat}_Y(\omega)} \setminus \bigcup_{\omega' < \omega} \overline{\mathrm{strat}_Y(\omega')}.$$

By Lemma 3.1.8 the strata $\mathrm{strat}(\omega)$ of the stratification associated to the embedded semifan $\Delta^\sigma_{\mathrm{stab}} \subset \Delta^\sigma$ satisfy the condition $\overline{\mathrm{strat}(\omega)} = \mathrm{Z}(\mathrm{val}(v_\omega), X_{\Delta^\sigma})$, $\mathrm{strat}(\omega) := \overline{\mathrm{strat}(\omega)} \setminus \bigcup_{\omega' < \omega} \mathrm{strat}(\omega')$. It follows by the above that the sets r define stratification R satisfying conditions (1), (2) and (3).

(4) Since the strata of R on Y are defined by centers of X-stable valuations, conditions (a), (b) and (c) are satisfied.

Let Q be a stratification on Y satisfying conditions (4a), (4b) and (4c).

For the morphism ϕ_σ^f induced by a chart ϕ_σ and for any $x \in U_\sigma$, let $\widetilde{\phi}_\sigma^f : Y \times_X \widehat{X}_x \to \widetilde{X}_{\Delta^\sigma}$ be a $G^\sigma \simeq \mathrm{Aut}(\widehat{X}_x, S)^0$-equivariant isomorphism of formal completions mapping the strata of Q isomorphically onto the strata of $\widetilde{S}_{\Omega^\sigma}$ on $\widetilde{X}_{\Delta^\sigma}$ defined by the embedded semifan $\Omega^\sigma \subset \widetilde{\Delta}^\sigma$

By (4a) the strata of $\widetilde{S}_{\Omega^\sigma}$ on $\widetilde{X}_{\Delta^\sigma}$ are G^σ-invariant. Hence for any two isomorphisms $\widetilde{\phi}_{\sigma,i}^f : Y \times_X \widehat{X}_x \to \widetilde{X}_{\Delta^\sigma}$, where $i = 1, 2$, induced by charts $\phi_{\sigma,i}$ the induced automorphism $\widetilde{\phi}_{\sigma,1}^f(\widetilde{\phi}_{\sigma,2}^f)^{-1} \in G^\sigma$ of $\widetilde{X}_{\Delta^\sigma}$ maps $\widetilde{S}_{\Omega_1^\sigma}$ to $\widetilde{S}_{\Omega_2^\sigma}$. Since both stratifications are G^σ-invariant we get $\widetilde{S}_{\Omega_1^\sigma} = \widetilde{S}_{\Omega_2^\sigma}$, $\Omega_1^\sigma = \Omega_2^\sigma$. Hence Ω^σ does not depend upon a chart. Consequently, if $\tau \le \sigma$ then $\Omega^\sigma|\tau = i_\tau^\sigma(\Omega^\tau)$. By condition (4a), the relative interior of a face $\omega \in \Omega^\sigma$ is contained in the interior of a face $\tau \le \sigma$. Moreover the closure $O_{i_\tau^\varrho(\omega)} \subset \widetilde{X}_{\Omega^\varrho}$ is G^ϱ-invariant for any $\varrho \ge \tau$. This shows that all faces $\omega \in \Omega^\sigma$ are Σ-stable. Consequently, $\Omega^\sigma \subset \Delta^\sigma_{\mathrm{stab}}$ and finally by Lemma 3.1.9. the corresponding stratification S_{Δ^σ} is finer than S_{Ω^σ}. $\qquad\square$

6.6. **Correspondence between toroidal morphisms and canonical subdivisions.** Proposition 6.5.3 can be rephrased as follows:

Theorem 6.6.1. *Let (X, S) be an oriented stratified toroidal variety (resp. Γ-stratified toroidal variety) with the associated oriented semicomplex (resp. Γ-semicomplex) Σ. There is a $1 - 1$ correspondence between the toroidal morphisms of stratified (resp. Γ-stratified) toroidal varieties $f : (Y, R) \to (X, S)$ and canonical subdivisions Δ of the oriented semicomplex (resp. orineted Γ-semicomplex) Σ. Moreover the semicomplex (resp. Γ-semicomplex) associated to (Y, R) is given by $\Sigma_R = \Delta_{\mathrm{stab}}$.* $\qquad\square$

Remark. In particular, if (X, S) is a toroidal embedding (see Lemma 6.3.1) then Σ is a complex and all its subdivisions are canonical (Lemma 6.3.1). We get a 1-1-correspondence between the subdivisions of the complex Σ and the toroidal morphisms $(Y, R) \to (X, S)$ (see [39]).

Example 6.6.2. Let X be a variety with isolated singularity of type $x_1 x_2 = x_3 x_4$ with the stratification consisting of the singular point and its complement (as in Example 4.10.7). Then the associated semicomplex Σ consists of the cone over a square and its vertex. As follows from Example 13.3.1 the stable support consists of the ray over the center of this square. Consequently, by Proposition 7.6.1 there are three canonical nontrivial subdivisions of Σ. One is the star subdivision at the stable ray corresponding to the blow-up of the point. Then Δ_{stab} consists of the ray and its vertex and corresponds to the toroidal embedding defined by the exceptional divisor and its complement. The other two are subdivisions defined by diagonals corresponding to two small resolutions of singularities. Δ_{stab} consists of the cone over the diagonal and its vertex and corresponds to the smooth stratified toroidal variety with the stratification defined by the exceptional curve-the preimage of the singular point and its complement.

The language of stratified toroidal varieties allows a combinatorial description of the <u>Hironaka twist</u>.

Example 6.6.3. ([29]) Let $X = \mathbf{P}^3$ be a projective 3-space containing two curves l_1 and l_2 intersecting transversally in two points p_1 and p_2. These data define a stratified toroidal variety (X, S). The associated semicomplex Σ consists of two 3-dimensional cones σ_1 and σ_2 corresponding respectively to the points p_1 and p_2, and sharing two 2-dimensional faces τ_1 and τ_2 corresponding to l_1 and l_2. The Hironaka' twist Y is obtained by glueing the consecutive blow-ups of $X \setminus \{p_i\}$ at l_1 and l_2 taken in two different orders, along the isomorphic open subsets over $X \setminus \{p_1\} \setminus \{p_2\}$. Then Y is a stratified toroidal variety. The preimage of p_i

consists of two irreducible curves l_{i1} and l_{i2} intersecting at p_{i0}. The stratification T on Y is determined by the above four curves, two points and two exceptional divisors-preimages of curves. Let v_i denote the sum of the two generators of τ_i. Let v^i denote the sum of the generators of σ_i.

Δ is the subdivision of Σ obtained by glueing the consecutive star subdivisions of σ_i at $\langle v_1 \rangle$ and $\langle v_2 \rangle$ taken in two different orders, along the star subdivisions of τ_i at $\langle v_i \rangle$. It follows from Example 13.3.2 that $\mathrm{Stab}(\Sigma)$ is the union of the cones $\langle v_1, v_2, v^1 \rangle \subset \sigma_1$ and $\langle v_1, v_2, v^2 \rangle \subset \sigma_2$. Then Δ_{stab} consists of the relevant cones in Δ whose relative interior intersects $\mathrm{Stab}(\Sigma)$. These cones correspond to the above mentioned strata on Y.

7. CANONICAL SUBDIVISIONS AND STABLE SUPPORT

7.1. Isomorphisms of local rings and linear transformations of stable supports.

Lemma 7.1.1.   1. *Let σ and τ be two cones of the same dimension. Let $\psi : \widehat{X}_\sigma \simeq \widehat{X}_\tau$ be an isomorphism preserving the closures of the toric orbits. Then there is a linear isomorphism $L_\psi : \sigma \to \tau$ such that $\psi_*(\mathrm{val}(v)) = \mathrm{val}(L_\psi(v))$ for any $v \in \sigma$.*
  2. *Let σ and τ be two Γ-semicones of the same dimension. Let $\psi : \widetilde{X}_\sigma \simeq \widetilde{X}_\tau$ be a Γ-equivariant isomorphism preserving strata. Then there is a linear isomorphism $L_\psi : \mathrm{Inv}(\sigma) \to \mathrm{Inv}(\sigma')$ such that $\psi_*(\mathrm{val}(v)) = \mathrm{val}(L_\psi(v))$ for any $v \in \mathrm{Inv}(\sigma)$.*

Proof.

(1) The isomorphism ψ maps the divisors of the orbit stratification on $\widetilde{X}_\sigma$ to the divisors of the orbit stratification on $\widetilde{X}_\tau$. Consequently, the local semiinvariant parameters at O_σ are mapped to local parameters at O_τ which differ from semiinvariant ones by invertible functions. This defines a linear isomorphism of cones $L : \sigma \to \tau$ mapping faces of σ corresponding to strata to suitable faces of τ Let $\psi_L : \widehat{X}_\sigma \simeq \widehat{X}_\tau$ denote the induced morphism. Then $\phi := \psi_L^{-1}\psi$ is an automorphism of $\widehat{X}_\sigma$ preserving strata. By Lemma 6.3.1 the automorphism ϕ preserves all valuations, hence $\psi_*(\mathrm{val}(v)) = \psi_{L*}(\mathrm{val}(v)) = \mathrm{val}(L(v))$.

(2) Let $v_1, v_2 \in \mathrm{Inv}(\sigma)$. Then $\psi_*(\mathrm{val}(v_1))$ and $\psi_*(\mathrm{val}(v_2))$ are G^τ-invariant, and correspond to $v_1', v_2' \in \mathrm{Inv}(\tau)$. Set $\Delta^\sigma = \langle v_2 \rangle \langle v_1 \rangle \cdot \sigma$ and $\Delta^\tau = \langle v_2' \rangle \langle v_1' \rangle \cdot \tau$.

The isomorphism $\psi : \widetilde{X}_\sigma \simeq \widetilde{X}_\tau$ lifts to an isomorphism $\Psi : \widetilde{X}_{\Delta^\sigma} \simeq \widetilde{X}_{\Delta^\tau}$. Let D_1 and D_2 (resp. D_1' and D_2') denote the exceptional divisors on $\widetilde{X}_{\Delta^\sigma}$ (resp. $\widetilde{X}_{\Delta^\tau}$) corresponding to v_1 and v_2 (resp. v_1' and v_2') The generic point of intersection $p := D_1 \cap D_2$ is mapped to the point of intersection $p' := D_1' \cap D_2'$. Set $\delta_1 := \langle v_1, v_2 \rangle \in \Delta^\sigma$ and $\delta_2 := \langle v_1', v_2' \rangle \in \Delta^\tau$. Then $(\widetilde{X}_{\delta_1}, \{D_1, D_2, p\})$ is a formal completion of a toroidal embedding which is mapped to $(\widetilde{X}_{\delta_2}, \{D_1', D_2', p'\})$.

By (1) Ψ_* determines a linear transformation on $\langle v_1, v_2 \rangle$ and consequently a linear transformation of $\mathrm{Inv}(\sigma)$ which is a linear isomorphism.   □

Definition 7.1.2. Let σ be a Γ-semicone. A vector $v_\sigma \in \mathrm{int}(\widetilde{\sigma})$ is *absolutely invariant* if $\mathrm{val}(v_\sigma)$ is invariant with respect to $\mathrm{Aut}(\widetilde{X}_\sigma)$.

Lemma 7.1.3. *Let σ be a Γ-semicone. There exists an absolutely invariant vector $v \in \mathrm{int}(\widetilde{\sigma})$.*

Proof. We can assume that $\sigma = \widetilde{\sigma}$ replacing σ, if necessary, by the Γ-semicone $\widetilde{\sigma} := \sigma \cup |\widetilde{\sigma}|$. Let $v \in \mathrm{int}(\widetilde{\sigma})$ be a minimal internal vector which by Lemma 5.3.16(3) is a stable vector. By Lemma 5.3.8 for any $g \in \mathrm{Aut}(\widetilde{X}_\sigma)$, $g_*(\mathrm{val}(v)) = \mathrm{val}(v')$, where v', is also minimal. The set W of minimal vectors is contained in $\overline{\mathrm{par}}(\sigma)$ and therefore it is finite. By the above, $\mathrm{Aut}(\widetilde{X}_\sigma)$ acts on W. Let v_σ denote the sum of all the minimal vectors. By Lemma 7.1.1, $\mathrm{Aut}(\widetilde{X}_\sigma)$ acts on $\mathrm{Inv}(\sigma) \supset W$ and therefore it acts trivially on v_σ.   □

7.2. Initial terms defined by a monomial valuation.

Lemma 7.2.1. *Let $F(x_1, \ldots, x_k) \in K[x_1, \ldots, x_k]$ be a quasihomogeneous polynomial of weight $w(F)$ with respect to weights $w(x_1), \ldots, w(x_k)$. Let ν be a valuation of a ring R and $u_1, \ldots, u_k \in R$ be such that $\nu(u_i) \geq w(x_i)$. Then $\nu(F(u_1, \ldots, u_k)) \geq w(F)$. If F belongs to the ideal $(x_l, \ldots, x_k)$ and $\nu(u_i) > w(x_i)$ for $i = l, \ldots, k$ then $\nu(F(u_1, \ldots, u_k)) > w(F)$.*

Proof. The proof can be reduced to the situation, in which case F is a monomial when it immediately follows from the definition of valuation.   □

Definition 7.2.2. Let ν be a monomial valuation of $K[[x_1, \ldots, x_k]]$ with a basis $x_1, \ldots, x_k$ and weights $a_1, \ldots, a_k > 0$. For $f \in K[[x_1, \ldots, x_k]]$ let $f = f_0 + f_1 + \ldots$ be a decomposition of f into an infinite sum of quasihomogeneous polynomials such that $\nu(f_0) < \nu(f_1) < \ldots$. Then by the *initial term* of f we mean $\mathrm{in}_\nu(f) = f_0$. Let $G = (G_1, \ldots, G_k)$ be an endomorphism of $K[[x_1, \ldots, x_k]]$ such that $x_i \mapsto G_i$. Then

$$\mathrm{in}_\nu(G) := (\mathrm{in}_\nu(G_1), \ldots, \mathrm{in}_\nu(G_k))$$

is the endomorphism determined by the initial terms of G_i.

Lemma 7.2.3. *Let $G = (G_1, \ldots, G_k)$ be an automorphism of $K[[x_1, \ldots, x_k]]$ preserving a monomial valuation ν: $G_*(\nu) = \nu$. Then $\mathrm{in}_\nu(G)$ is an automorphism of $K[[x_1, \ldots, x_k]]$ preserving ν. Moreover for any $f \in K[[x_1, \ldots, x_k]]$ we have $\mathrm{in}_\nu(G(f)) = \mathrm{in}_\nu(G)(\mathrm{in}_\nu(f))$.*

Proof. Write $G_i = \mathrm{in}_\nu(G_i) + G_{ih}$, where $\nu(G_{ih}) > \nu(G_i)$.

For any $f \in K[[x_1, \ldots, x_k]]$ write $f = \mathrm{in}_\nu(f) + f_h$, where $\nu(f_h) > \nu(f)$. Then we have $G(f) = f(G_1, \ldots, G_n) = \mathrm{in}_\nu f(G_1, \ldots, G_n) + f_h(G_1, \ldots, G_n)$, where by Lemma 7.2.1, $\nu(f_h(G_1, \ldots, G_n)) > \nu(f)$.

$\mathrm{in}_\nu(f)(x_1 + y_1, \ldots, x_k + y_k)$ is a quasihomogeneous polynomial in $x_1, \ldots, x_k, y_1, \ldots, y_k$ with weights $\nu(x_1) = \nu(y_1), \ldots, \nu(x_k) = \nu(y_k)$. Write $\mathrm{in}_\nu f(x_1 + y_1, \ldots, x_k + y_k) = \mathrm{in}_\nu(f)(x_1, \ldots, x_n) + f_h(x_1, y_1, \ldots, x_n, y_n)$ where f_h is a quasihomogeneous polynomial in the ideal $(y_1, \ldots, y_n) \cdot K[[x_1, \ldots, x_k, y_1, \ldots, y_k]]$. Applying Lemma 7.2.1 to $x_i = \mathrm{in}_\nu(G_i)$ and , $y_i = G_{ih}$ we obtain

$$(\mathrm{in}_\nu(f)(G_1, \ldots, G_n)) = \mathrm{in}_\nu(f)(\mathrm{in}_\nu(G_1) + G_{1h}, \ldots, \mathrm{in}_\nu(G_n) + G_{nh})) =$$
$$\mathrm{in}_\nu(f)(\mathrm{in}_\nu(G_1), \ldots, \mathrm{in}_\nu(G_n))) + f_h(\mathrm{in}_\nu(G_1), \ldots, \mathrm{in}_\nu(G_n), G_{1h}, \ldots, G_{nh})),$$

where by Lemma 7.2.1, $\nu(f_h(\mathrm{in}_\nu(G_1), \ldots, \mathrm{in}_\nu(G_n), G_{1h}, \ldots, G_{nh})) > \nu(f)$, which gives

$$\mathrm{in}_\nu(G(f)) = \mathrm{in}_\nu(G)(\mathrm{in}_\nu(f)).$$

Now let $G^{-1} = (G'_1, \ldots, G'_n)$. Then by the above,

$$G(x_i) \circ G^{-1}(x_i) = G'_i(G_1, \ldots, G_n) = x_i \quad \text{and}$$
$$\mathrm{in}_\nu(G'_i)(\mathrm{in}_\nu(G_1), \ldots, \mathrm{in}_\nu(G_n)) = x_i.$$

This shows that $\mathrm{in}_\nu(G)$ and $\mathrm{in}_\nu(G^{-1})$ are automorphisms. They are defined by quasihomogeneous polynomials of ν-degrees $\nu(x_1), \ldots, \nu(x_k)$. It follows from Lemma 7.2.1 that $\nu(\mathrm{in}_\nu(G(f))) \geq \nu(f)$ and $\nu(\mathrm{in}_\nu(G^{-1}(f))) \geq \nu(f)$ for any $f \in K[[x_1, \ldots, x_k]]$. Consequently, $\nu(\mathrm{in}_\nu(G(f))) = \nu(f)$. $\square$

Lemma 7.2.4. *Let $u_1, \ldots, u_n$ be local semiinvariant parameters on $\widetilde{X}_\sigma$. Write $K[\widetilde{X}_\sigma] = K[[x_1, \ldots, x_k]]/I$. Let ν be a toric valuation on $\widetilde{X}_\sigma$ invariant with respect to the group of all automorphisms of $\widetilde{X}_\sigma$ centered at $O_{\widetilde{\sigma}}$. Then ν defines a monomial valuation on $K[[x_1, \ldots, x_k]]$ with basis $x_1, \ldots, x_n$ and weights $\nu(u_1), \ldots, \nu(u_n)$. Moreover:*

1. *Any automorphism g of $K[\widetilde{X}_\sigma]$ extends to an automorphism G of $K[[x_1, \ldots, x_k]]$ preserving ν and I.*
2. *The automorphism $\mathrm{in}_\nu(G)$ of $K[[x_1, \ldots, x_k]]$ defines an automorphism $\mathrm{in}_\nu(g)$ of $K[\widetilde{X}_\sigma]$ preserving ν.*

Proof. Any automorphism g of $K[\widetilde{X}_\sigma]$ sends parameters $u_1, \ldots, u_n$ to $g_1 = g^*(u_1), \ldots, g_n = g^*(u_n)$. Moreover $\nu(g_i) = \nu(u_i)$ so we may assume that $g_i = G_i(u_1, \ldots, u_n)$ are given by some formal power series $G_i \in K[[x_1, \ldots, x_k]]$ of ν-weights $\nu(u_i)$. Let $h = g^{-1}$, and let $h_1 = h^*(u_1), \ldots, h_n = h^*(u_n)$ be given by some formal power series $H_i \in K[[x_1, \ldots, x_k]]$ of ν-weights $\nu(u_i)$. Then $\nu(H_i(G_1, \ldots, G_k)) \geq \nu(u_i)$. On the other hand $H_i(G_1(u_1, \ldots, u_n), \ldots, G_k(u_1, \ldots, u_n)) = u_i$. Hence $(H_i(G_1, \ldots, G_k) - x_i) \in I$. In other words $H_i(G_1, \ldots, G_k) = x_i + F_i(x_1, \ldots, x_k)$, where $\nu(F_i) \geq \nu(u_i)$ and $F_i \in I$. Consequently, for any $F \in K[[x_1, \ldots, x_k]]$, $\nu(F) \geq \nu(F(G_1, \ldots, G_k))$.

Let $m \subset K[[x_1, \ldots, x_k]]$ be the maximal ideal. By definition $m/(m^2 + I) = m/m^2$ and thus $I \subset m^2$. This gives $F_i \in m^2$. Then the ring automorphism $\Phi := H \circ G$ of $K[[x_1, \ldots, x_k]]$ can be written as a K-linear transformation $\Phi = \mathrm{id} + F$, where $F(K[[x_1, \ldots, x_k]]) \subset I$ and $F(m^i) \subset m^{i+1}$. It follows from Lemma 7.2.1 that for any formal power series $x \in K[[x_1, \ldots, x_k]]$, $\nu(\Phi(x)) \geq \nu(x)$ and hence $\nu(F(x)) \geq \min\{\nu(\phi(x)), \nu(x)\} = \nu(x)$. The inverse of Φ can be written as $\Phi^{-1} = \mathrm{id} + F'$, where $F' = -F + F^2 - F^3 + \ldots$. By the above $F'(K[[x_1, \ldots, x_k]]) \subset I$ and $\nu(F'(x)) \geq \nu(x)$, which shows that $\nu(\Phi^{-1}(x)) \geq \nu(x)$ and finally $\nu(\Phi^{-1}(x)) = \nu(x)$. Hence G and $G^{-1} = \Phi^{-1} \circ H$ are automorphisms satisfying $\nu(G^{-1}(x) \geq \nu(x)$. By Lemma 7.2.1 and the above, $\nu(G(x)) \geq \nu(x)$ for any $x \in K[[x_1, \ldots, x_k]]$. Hence $\nu(G(x)) = \nu(x)$. We have shown that any automorphism g of $K[\widetilde{X}_\sigma]$ is defined by an automorphism G of $K[[x_1, \ldots, x_k]]$ preserving the valuation ν and the ideal I. Then by Lemma 7.2.3, $\mathrm{in}_\nu(G)$ is an automorphism of $K[[x_1, \ldots, x_k]]$ preserving the valuation ν and the ideal $\mathrm{in}_\nu(I) = I$. Therefore it defines an automorphism $\mathrm{in}_\nu(G)$ of $K[\widetilde{X}_\sigma]$. $\square$

7.3. **Structure of the group of automorphisms of the completion of a local ring of a toric variety.**

Lemma 7.3.1. *Let σ be a Γ-semicone. Let ν be the valuation on $\widetilde{X}_\sigma$ associated to the absolutely invariant vector v_σ. Let $G_\nu^\sigma := \mathrm{Aut}_\nu(\widetilde{X}_\sigma)$ denote the group of all quasihomogeneous automorphisms with respect to semiinvariant coordinates on $\widetilde{X}_\sigma$ with weights detrmined by ν. Then*

1. *G_ν^σ is an algebraic group.*
2. *There is a surjective morphism $\Phi : \mathrm{Aut}(\widetilde{X}_\sigma) \to \mathrm{Aut}_\nu(\widetilde{X}_\sigma)$ defined by taking the lowest degree quasihomogeneous part.*
3. *The kernel $\ker_\sigma$ of Φ is a connected proalgebraic group.*
4. *$\mathrm{Aut}(\widetilde{X}_\sigma) = G_\nu^\sigma \cdot \ker_\sigma$.*
5. *For any $g \in \ker_\sigma$ there exists a morphism $i_g : \mathbf{A}^1 \to \ker_\sigma$ such that $g = i_g(1)$, $\mathrm{id} = i_g(0)$.*
6. *For any $g \in \mathrm{Aut}(\widetilde{X}_\sigma)^0$ there exists a morphism $i_g : W := \mathbf{A}^1 \times (G_\nu^\sigma)^0 \to \mathrm{Aut}(\widetilde{X}_\sigma)^0$ such that $\{g, \mathrm{id}\} \subset i_g(W)$.*

Proof. Let $u_1, \ldots, u_k$ be semiinvariant parameters on $\widetilde{X}_\sigma$ generating the maximal ideal of $O_{\widetilde{\sigma}}$. Denote by $m_1, \ldots, m_k \in \widetilde{\sigma}$ the corresponding characters.

(1) Let $\mathrm{Aut}_\nu(\mathbf{A}^k)$ denote the group of all quasihomogeneous automorphisms of $\mathbf{A}^k$. By Lemma 7.2.4, $\mathrm{Aut}_\nu(\widetilde{X}_\sigma)$ is isomorphic to the subgroup of $\mathrm{Aut}_\nu(\mathbf{A}^k)$ consisting of the automorphisms preserving I. The automorphisms of $\mathrm{Aut}_\nu(\mathbf{A}^k)$ are described by sets of k quasihomogeneous polynomials with weights $\nu(u_1), \ldots, \nu(u_k)$ and having linearly independent linear terms. Thus $\mathrm{Aut}_\nu(\mathbf{A}^k)$ and $\mathrm{Aut}_\nu(\widetilde{X}_\sigma)$ are algebraic.

(2) This follows from Lemma 7.2.1 and 7.2.4

(3)(4)(5) The 1-parameter subgroup $t \mapsto t^{\nu_\sigma}$ defines the action on $\widetilde{X}_\sigma$ such that $\psi_t(u_i) = t^{(m_i, v_\sigma)} u_i = t^{\nu(u_i)} u_i = t^{\alpha_i}$, where $\alpha_i = \nu(u_i)$ is a ν-weight of u_i. The action defines an embedding $K^* < \mathrm{Aut}(\widetilde{X}_\sigma)$.

Write $g \in \ker_\sigma$ in terms of coordinate functions, $g = (g_1, \ldots, g_k)$, where $g_i = g^*(u_i)$. Let $\alpha_{i,0} = \alpha_i$ denote the ν-weight of g_i and let $g_i = u_i + g_{\alpha_{i,1}} + \ldots + g_{\alpha_{i,j}} + \ldots$ be the decomposition according to ν-weights $\alpha_{i,0} < \alpha_{i,1} < \ldots < \alpha_{i,j} < \ldots$. Define the morphism $i_g : K^* \to \ker_\sigma$ by $i_g(t) := \psi_t^{-1} \circ g \circ \psi_t$. Then $(i_g(t))^*(u_i) = (\psi_t^{-1} \circ g \circ \psi_t)^*(u_i) = (g \circ \psi_t)^* t^{-\alpha_i}(u_i) = \psi_t^*(t^{-\alpha_i} g_i) = t^{-\alpha_i}(t^{\alpha_{i,0}} u_i + t^{\alpha_{i,1}} g_{\alpha_{i,1}} + \ldots + t^{\alpha_{i,j}} g_{\alpha_{i,j}} + \ldots) == u_i + t^{\alpha_{i,1} - \alpha_{i,0}} g_{\alpha_{i,1}} + \ldots + t^{\alpha_{i,j} - \alpha_{i,0}} g_{\alpha_{i,j}} + \ldots$. The above morphism extends to a morphism $i_g : \mathbf{A}^1 \to \ker_\sigma$ "connecting" id to g. This shows that $\ker_\sigma$ is connected. Note that Ψ maps the connected component $\mathrm{Aut}_\nu(\widetilde{X}_\sigma)^0$ to the connected component $\mathrm{Aut}(\widetilde{X}_\sigma)^0$.

(6) Write $g \in \mathrm{Aut}(\widetilde{X}_\sigma, S)^0$ as $g = g_1 g_2$, where $g_1 \in \ker_\sigma$, $g_2 \in G_\nu^\sigma$. Then for $t \in \mathbf{A}^1$ and $h \in G_\nu^\sigma$, set $i_g(t, h) := i_{g_1}(t) \cdot h$. $\square$

Lemma 7.3.2. *([22])) Let σ be a Γ-semicone. Then $\mathrm{Aut}(\widetilde{X}_\sigma)^0 \subset \mathrm{Aut}(\widetilde{X}_\sigma)$ is a normal subgroup and there is a natural surjection $\mathrm{Aut}(\sigma) \longrightarrow \mathrm{Aut}(\widetilde{X}_\sigma)/\mathrm{Aut}(\widetilde{X}_\sigma)^0$.*

Proof. Set $G := \mathrm{Aut}_\nu(\widetilde{X}_\sigma) = \mathrm{Aut}_\nu(X_{\widetilde{\sigma}})$ (notation of Lemma 7.3.1). It follows from Lemma 7.3.1 that the natural inclusion $G \subset \mathrm{Aut}(\widetilde{X}_\sigma)$ determines a group isomorphism $G/G^0 \simeq \mathrm{Aut}(\widetilde{X}_\sigma)/(\mathrm{Aut}(\widetilde{X}_\sigma))^0$. Then we get a surjection $N_G(T)/T \to G/G^0$. Let $H := \mathrm{Aut}_T(X_\sigma)$ be the group of the Γ-equivariant automorphisms of X_σ, preserving the big torus T. It suffices to show that $N_G(T) = H$ and $H/T \simeq \mathrm{Aut}(\sigma)$. If $g \in N_G(T)$ then $gTg^{-1} = T$. Let $x \in T \subset X_{\widetilde{\sigma}}$. Then $gTg^{-1}x = Tx$ or equivalently $Tg^{-1}x = g^{-1}Tx$. Since the latter subset is open, $g^{-1}x$ is in the big open orbit $Tg^{-1}x = Tx$, in other words, $g^{-1}x \in T$ which shows that $g \in H$ and $N_G(T) \subset H$. Let H^0 be the connected component of H containing id. Then H_0 acts trivially on irreducible components of the complement of T, that is, on T-invariant Weil and Cartier divisors. Hence it multiplies characters by invertible functions on $X_{\widetilde{\sigma}}$. Invertible functions on T are monomials. Invertible monomials on $X_{\widetilde{\sigma}}$ are constants. Thus H_0 acts on characters multiplying them by nonzero constants. This shows that $H_0 = T$. Hence for any $g \in H$, $gTg^{-1} = T$, which shows that $N_G(T) \supset H$. H/T can be identified with the subgroup of H preserving $1 \in T \subset X_{\widetilde{\sigma}}$, that is, the automorphisms mapping characters to characters. The latter subgroup is equal to the group $\mathrm{Aut}(\widetilde{\sigma})$ of automorphisms of embedded semifan $\sigma \subset \widetilde{\sigma}$.

The restriction of an automorphism in $\mathrm{Aut}(\widetilde{\sigma})$ is an automorphism of σ. Any automorphism of σ defines an automorphism of $\widetilde{\sigma} = \sigma \times \mathrm{reg}(\sigma)$. Thus there are natural group homomorphisms $i : \mathrm{Aut}(\sigma) \to \mathrm{Aut}(\widetilde{\sigma})$ and $p : \mathrm{Aut}(\widetilde{\sigma}) \to \mathrm{Aut}(\sigma)$ such that $p \circ i = \mathrm{id}_\sigma$. Moreover p corresponds to the restriction of automorphisms of $X_{\widetilde{\sigma}}$ to the subvariety $X_\sigma \times O_{\mathrm{reg}(\sigma)} \subset X_{\widetilde{\sigma}}$. By Lemma 4.9.14, $p(\mathrm{Aut}(\widetilde{\sigma})^0 = \mathrm{Aut}(\sigma)^0$. $\square$

7.4. **Invariant and semiinvariant valuations.**

Lemma 7.4.1. *Let $v \in \sigma$ be an integral vector. Then $\operatorname{val}(v)$ is an $\operatorname{Aut}(\widetilde{X}_\sigma)^0$-invariant valuation on $\widetilde{X}_\sigma$ iff it is $\operatorname{Aut}(\widetilde{X}_\sigma)$-semiinvariant.*

Proof. ($\Leftarrow$). If $\operatorname{val}(v)$ is $\operatorname{Aut}(\widetilde{X}_\sigma)$-semiinvariant then it is $\operatorname{Aut}(\widetilde{X}_\sigma)^0$-semiinvariant. Hence by Lemma 5.3.15(1) it is $\operatorname{Aut}(\widetilde{X}_\sigma)^0$-invariant. ($\Rightarrow$). If $\operatorname{val}(v)$ is $\operatorname{Aut}(\widetilde{X}_\sigma)$-invariant then by Lemma 7.3.2, $\operatorname{Aut}(\widetilde{X}_\sigma)$ acts on $\operatorname{val}(v)$ as the finite group $\operatorname{Aut}(\sigma) \longrightarrow \operatorname{Aut}(\widetilde{X}_\sigma)/\operatorname{Aut}(\widetilde{X}_\sigma)^0$. Therefore $\operatorname{val}(v)$ is $\operatorname{Aut}(\widetilde{X}_\sigma)$-semiinvariant. $\square$

Lemma 7.4.2. *Let L be any algebraically closed field containing K. Then $\operatorname{val}(v)$ defines $\operatorname{Aut}(\widetilde{X}_\sigma)^0$-invariant valuation on $\widetilde{X}_\sigma$ iff it defines an $\operatorname{Aut}_L(\widetilde{X}_\sigma)^0$-invariant valuation on $\widetilde{X}_\sigma^L$.*

Proof. Follows from Lemma 4.5.3(3). $\square$

For a Γ-semicone σ denote by $\widehat{\operatorname{Inv}}(\sigma)$ the cone generated by vectors $v \in |\sigma|$, for which valuations $\operatorname{val}(v, \widehat{X}_\sigma)$ are invariant with respect to $\operatorname{Aut}(\widehat{X}_\sigma)^0$.

Lemma 7.4.3. *For a Γ-semicone σ the two cones are equal $\widehat{\operatorname{Inv}}(\sigma) = \operatorname{Inv}(\sigma)$.*

Proof. Note that $K[\widetilde{X}_\sigma] = L[\widehat{X}_\sigma]$ for suitable $L := K[[x_1, \ldots, x_k]]$. Let (L) denote the algebraic closere of the quotient field of L. Each automorphism of $K[[\sigma^\vee]]$ defines the automorphism of $L[[\sigma^\vee]]$ which is constant on L. Each automorphism in $^\vee\operatorname{Aut}(\widehat{X}_\sigma)^0$ determines an automorphism in $\operatorname{Aut}(\widetilde{X}_\sigma)^0$, so that there is a monomorphism of proalgebraic groups $\imath : \operatorname{Aut}(\widehat{X}_\sigma)^0 \to \operatorname{Aut}(\widetilde{X}_\sigma)^0$. Therefore if $\operatorname{val}(v, \widetilde{X}_\sigma)$ is invariant with respect to $\operatorname{Aut}(\widetilde{X}_\sigma)^0$ then it is $\imath(\operatorname{Aut}(\widehat{X}_\sigma)^0)$-invariant. Since the subring $K[\widehat{X}_\sigma] \subset K[\widetilde{X}_\sigma]$ is $\imath(\operatorname{Aut}(\widehat{X}_\sigma)^0)$-invariant then the restriction $\operatorname{val}(v, \widehat{X}_\sigma)$ of $\operatorname{val}(v, \widetilde{X}_\sigma)$ to $\widehat{X}_\sigma$ is $\operatorname{Aut}(\widehat{X}_\sigma)^0$-invariant.

Now, if $\operatorname{val}(v, \widehat{X}_\sigma)$ is $\operatorname{Aut}(\widehat{X}_\sigma)^0$-invariant then by Lemma 7.4.2, it defines an $\operatorname{Aut}_{(L)}(\widehat{X}_\sigma^{(L)})^0$-invariant valuation. By Lemma 7.4.1, $\operatorname{val}(v, \widehat{X}_\sigma^{(L)})$ is $\operatorname{Aut}_{(L)}(\widehat{X}_\sigma^{(L)})$-semiinvariant. The proalgebraic group $\operatorname{Aut}_L(\widetilde{X}_\sigma) = \operatorname{Aut}_L(\widehat{X}_\sigma^L)$ is a subgroup (as an abstract group) of $\operatorname{Aut}_{(L)}(\widehat{X}_\sigma^{(L)})$. Thus the restriction $\operatorname{val}(v, \widetilde{X}_\sigma)$ of $\operatorname{val}(v, \widehat{X}_\sigma^{(L)})$ is $\operatorname{Aut}_L(\widetilde{X}_\sigma)$-semiinvariant. By Lemma 4.7.1 any automorphism ϕ in $\operatorname{Aut}(\widetilde{X}_\sigma)$ can be decomposed as $\phi = \phi_0\phi_1$, where ϕ_0 preserves monomials and $\phi_1 \in \operatorname{Aut}_L(\widetilde{X}_\sigma)$. Therefore $\operatorname{val}(v, \widetilde{X}_\sigma)$ is $\operatorname{Aut}(\widetilde{X}_\sigma)$-semiinvariant and consequently by Lemma 7.4.1 it is $\operatorname{Aut}(\widetilde{X}_\sigma)^0$-invariant. $\square$

Lemma 7.4.4. *If $\sigma \le \tau$ then*

1. $\operatorname{Inv}(\sigma) \subset \operatorname{Inv}(\tau)$.
2. $\operatorname{stab}(\sigma) \subset \operatorname{stab}(\tau) \cap \sigma$.

Proof. (1) Let $v \in \operatorname{Inv}(\sigma)$. Any automorphism $\phi \in \operatorname{Aut}(\widetilde{X}_\tau)$ preserves the stratum $\operatorname{strat}(\sigma)$ and thus defines an automorphism $\widehat{\phi}$ of the completion of $Y := \widetilde{X}_\tau$ at $\operatorname{strat}(\sigma)$ which by Lemma 4.7.3 is isomorphic to the spectrum of the ring $K[\widehat{Y}_{\operatorname{strat}(\sigma)}] = K_\sigma[\widehat{X}_\sigma]$, where K_σ is the residue field of the generic orbit point O_σ. By Lemma 4.7.1 any automorphism $\widehat{\phi}$ of $K_\sigma[\widehat{X}_\sigma]$ can be written as $\widehat{\phi} = \phi_0\phi_1$, where ϕ_0 preserves monomials and $\phi_1 \in \operatorname{Aut}_{K_\sigma}(\widehat{X}_\sigma)$. By Lemma 7.4.3, $\operatorname{val}(v, \widehat{X}_\sigma)$ is $\operatorname{Aut}(\widehat{X}_\sigma)^0$-invariant. Let $\overline{K_\sigma}$ be the algebraic closure of K_σ. By Lemma 7.4.2, $\operatorname{val}(v, \widehat{X}_\sigma^{\overline{K_\sigma}})$ is $\operatorname{Aut}_{K_\sigma}(\widehat{X}_\sigma^{\overline{K_\sigma}})^0$-invariant. By Lemma 7.4.1, $\operatorname{val}(v, \widehat{X}_\sigma^{\overline{K_\sigma}})$ is $\operatorname{Aut}_{K_\sigma}(\widehat{X}_\sigma^{\overline{K_\sigma}})$-semiinvariant. Then its restriction $\operatorname{val}(v, \widehat{X}_\sigma^{K_\sigma})$ is $\operatorname{Aut}_{K_\sigma}(\widehat{X}_\sigma)$-semiinvariant. Thus $\phi_{1*}(\operatorname{val}(v, \widehat{X}_\sigma^{K_\sigma}))$ can be one of the finitely many toric valuations on $\widehat{X}_\sigma^{K_\sigma}$ for any $\phi_1 \in \operatorname{Aut}_{K_\sigma}(\widehat{X}_\sigma)$. But then $\widehat{\phi}_*(\operatorname{val}(v, \widehat{Y}_{\operatorname{strat}(\sigma)})) = \phi_{1*}(\operatorname{val}(v, \widehat{Y}_{\operatorname{strat}(\sigma)}))$ can be one of the finitely many toric valuations on $K[\widehat{Y}_{\operatorname{strat}(\sigma)}]$ for all $\phi \in \operatorname{Aut}(\widetilde{X}_\tau)$. The restrictions of these valuations to the local ring of $\widetilde{X}_\tau$ at $\operatorname{strat}(\sigma)$ define finitely many valuations on $\widetilde{X}_\tau$. Hence $\operatorname{val}(v, \widetilde{X}_\tau)$ is semiinvariant and consequently invariant on $\widetilde{X}_\tau$. (2) Follows from (1) and from Lemma 6.1.2. $\square$

7.5. **Group of divisor classes of the completion of a local ring of a toric variety.**

Lemma 7.5.1. *Let σ be a cone in a lattice N and Δ be a suddivision σ. Let $\widehat{X}_\Delta := X_\Delta \times_{X_\sigma} \widehat{X}_\sigma$. The following groups of divisor classes are isomorphic. (The isomorphisms are determined by the natural morphisms).*

1. $\operatorname{Cl}(\widehat{X}_\sigma) \simeq \operatorname{Cl}(X_\sigma)$, $\operatorname{Pic}(\widehat{X}_\sigma) \simeq \operatorname{Pic}(X_\sigma)$.
2. $\operatorname{Cl}(\widehat{X}_\Delta) \simeq \operatorname{Cl}(X_\Delta)$, $\operatorname{Pic}(\widehat{X}_\Delta) \simeq \operatorname{Pic}(X_\Delta)$.

3. *For the affine toric variety X_σ set $\mathbf{A}^1 \widehat{\times} \widehat{X}_\sigma := \lim_{\longrightarrow} \mathbf{A}^1 \times X_\sigma^{(n)} = \mathrm{Spec}(K[t][\widehat{X}_\sigma])$. Then*
$$\mathrm{Cl}(\mathbf{A}^1 \widehat{\times} \widehat{X}_\sigma) \simeq \mathrm{Cl}(\mathbf{A}^1 \times X_\sigma) \simeq \mathrm{Cl}(X_\sigma) \,,\; \mathrm{Pic}(\mathbf{A}^1 \widehat{\times} \widehat{X}_\sigma) \simeq \mathrm{Pic}(\mathbf{A}^1 \times X_\sigma) \simeq \mathrm{Pic}(X_\sigma).$$

4. *For any subdivision Δ of σ set $\mathbf{A}^1 \widehat{\times} \widehat{X}_\Delta := (\mathbf{A}^1 \widehat{\times} \widehat{X}_\sigma) \times_{X_\sigma} X_\Delta$. Then*
$$\mathrm{Cl}(\mathbf{A}^1 \widehat{\times} \widehat{X}_\Delta) \simeq \mathrm{Cl}(\mathbf{A}^1 \times \widehat{X}_\Delta) \simeq \mathrm{Cl}(X_\Delta),\; \mathrm{Pic}(\mathbf{A}^1 \widehat{\times} \widehat{X}_\Delta) \simeq \mathrm{Pic}(\mathbf{A}^1 \times \widehat{X}_\Delta) \simeq \mathrm{Pic}(X_\Delta).$$

Proof. Let $v_1, \ldots, v_k$ denote the generators of $\sigma \cap N_\sigma$ and let $\tau = \langle e_1, \ldots, e_k \rangle$ denote the regular k-dimensional cone. Let $\pi : \tau \to \sigma$ be the projection defined by $\pi(e_i) = v_i$. Then π defines a surjective morphism of lattices $\pi : N_\tau \to N_\sigma$ whose kernel is a saturated sublattice $N \subset N_\tau$. Thus the projection π defines the quotient map $X_\tau \to X_\sigma = X_\tau // T$ for the subtorus $T \subset T_\tau$ corresponding to the sublattice $N \subset N_\tau$. This also defines the quotient morphism $\alpha : \widehat{X}_\tau \to \widehat{X}_\sigma$, where $K[\widehat{X}_\sigma] = K[\widehat{X}_\tau]^T$.

Denote by $T_\tau \subset X_\tau$, $T_\sigma \subset X_\sigma$ the relevant tori and set $\widehat{T}_\tau := \widehat{X}_\tau \times_{X_\tau} T_\tau \subset \widehat{X}_\tau$, $\widehat{T}_\sigma := \widehat{X}_\sigma \times_{X_\sigma} T_\sigma \subset \widehat{X}_\sigma$. Then both schemes are nonsingular and let $\alpha' : \widehat{T}_\tau \to \widehat{T}_\sigma$ be the restriction of α. We also have $K[\widehat{T}_\sigma] = K[\widehat{T}_\tau]^T$.

The morphism α defines a group homomorphism $\mathrm{Cl}(\widehat{T}_\sigma) = \mathrm{Pic}(\widehat{T}_\sigma) \xrightarrow{\alpha^*} \mathrm{Pic}(\widehat{T}_\tau) = \mathrm{Cl}(\widehat{T}_\tau) = 0$. The last group is 0 since the ring $K[\widehat{X}_\tau]$ as well as its localization $K[\widehat{T}_\tau]$ are UFD. Let D be an effective Cartier divisor on $\widehat{T}_\sigma$. Then $\alpha^*(D)$ is T-invariant and principal on $\widehat{T}_\tau$. This means that the ideal I of $\alpha^*(D)$ is generated by $f \in K[\widehat{T}_\tau]$. By multiplying by a suitable monomial we can asume that $f \in K[\widehat{X}_\tau]^T$. Note that all $t \cdot f$, $t \in T$ also belong to I. Let M_T denote the lattice of characters of T. Let f_β denote the component of f with weight $\beta \in M_T$. By considering $K[\widehat{X}_\tau]/m^k$, where $m \subset K[\widehat{X}_\tau]$ is the maximal ideal, we see that all $f_\beta + m^k \in I \cdot K[\widehat{X}_\tau]/m^k$. Moreover $I \cdot K[\widehat{X}_\tau]/m^k$ is generated by all $f_\beta + m^k$. This yields $I = (f_\beta)_{\beta \in M_T}$ or $(1) = (f_\beta/f)$ showing that there is a β_0 such that f_{β_0}/f is invertible, which means $I = (f_{\beta_0})$. By multiplying by monomials we can assume that f_{β_0} is T-invariant. Thus $f_{\beta_0} \in K[\widehat{T}_\sigma]^T = K[\widehat{X}_\tau]$ and generates the ideal of D. But then D is principal on $\widehat{X}_\tau$ and finally $\mathrm{Cl}(\widehat{T}_\sigma) = 0$.

Now any divisor D in $\mathrm{Cl}(\widehat{X}_\sigma)$ (respectively in $\mathrm{Cl}(\widehat{X}_\Delta)$) is linearly equivalent to the T-invariant one $D' := D - (f_D)$, where $D_{|\widehat{T}_\sigma} = (f_D)_{|\widehat{T}_\sigma}$.

(3)(4) We repeat the reasoning from (1) and (2) and use the fact that $K[\mathbf{A}^1 \widehat{\times} \widehat{X}_\tau] = K[t][[x_1, \ldots, x_k]]$ is UFD (see Bourbaki[11]). $\square$

Lemma 7.5.2.   1. *G^σ acts trivially on $\mathrm{Cl}(\widetilde{X}_{\Delta^\sigma})$.*
   2. *Let Δ^σ be a subdivision of σ such that $\widetilde{X}_{\Delta^\sigma} \to \widetilde{X}_\sigma$ is G^σ-equivariant. Then G^σ acts trivially on $\mathrm{Cl}(\widetilde{X}_{\Delta^\sigma})$.*

Proof. We shall use the notation and results from Lemma 7.3.1.

(1) The natural morphism $\widetilde{X}_{\Delta^\sigma} \to X_{\Delta^\sigma}$ is G_ν^σ-equivariant. By Lemma 7.5.1 this morphism induces a G_ν^σ-equivariant isomorphism $\mathrm{Cl}(\widetilde{X}_\sigma) \to \mathrm{Cl}(X_\sigma)$. By Sumihiro [67] the algebraic linear group acts trivially on $\mathrm{Cl}(X_\sigma)$, which yields a trivial action on $\mathrm{Cl}(\widetilde{X}_\sigma)$. By Lemma 7.3.1 it suffices to show that ker_σ acts trivially on $\mathrm{Cl}(\widetilde{X}_\sigma)$.

Let $\widetilde{X}_\sigma^{\mathrm{ns}} \subset \widetilde{X}_\sigma$ denote the open subset of nonsingular points. Its complement is of codimension 2. We get a G^σ-equivariant isomorphism $\mathrm{Pic}(\widetilde{X}_\sigma^{\mathrm{ns}}) = \mathrm{Cl}(\widetilde{X}_\sigma^{\mathrm{ns}}) \simeq \mathrm{Cl}(\widetilde{X}_\sigma)$. It suffices to prove that ker_σ acts trivially on $\mathrm{Pic}(\widetilde{X}_\sigma^{\mathrm{ns}})$. Fix $g \in G$. Let $i_{g,\mathbf{A}^1} : \mathbf{A}^1 \to \mathrm{ker}_\sigma$ be the morphism from Lemma 7.3.1. Let $\Phi : G^\sigma \widehat{\times} \widetilde{X}_\sigma^{\mathrm{ns}} \to \widetilde{X}_\sigma^{\mathrm{ns}}$ be the action morphism from Lemma 4.5.1 and set $\Phi_{\mathbf{A}^1} := i_{g,\mathbf{A}^1} \circ \Phi : \mathbf{A}^1 \widehat{\times} \widetilde{X}_\sigma^{\mathrm{ns}} \to \widetilde{X}_\sigma^{\mathrm{ns}}$. Denote by $p : \mathbf{A}^1 \widehat{\times} \widetilde{X}_\sigma^{\mathrm{ns}} \to \widetilde{X}_\sigma^{\mathrm{ns}}$ the standard projection. By Lemma 7.5.1(4), $p^* : \mathrm{Pic}(\widetilde{X}_\sigma^{\mathrm{ns}}) \to \mathrm{Pic}(\mathbf{A}^1 \widehat{\times} \widetilde{X}_\sigma^{\mathrm{ns}})$ is an isomorphism . Let $D \in \mathrm{Pic}(\widetilde{X}_\sigma^{\mathrm{ns}})$. Let $j_g : \widetilde{X}_\sigma^{\mathrm{ns}} \to \{g\} \widehat{\times} \widetilde{X}_\sigma^{\mathrm{ns}} \subset \mathbf{A}^1 \widehat{\times} \widetilde{X}_\sigma^{\mathrm{ns}}$ denote the standard embedding. Then $p \circ j_g = \mathrm{id}_{|\widetilde{X}_\sigma^{\mathrm{ns}}}$. Since p^* is an isomorphism there is $D' \in \mathrm{Pic}(\widetilde{X}_\sigma^{\mathrm{ns}})$ such that $\Phi_{\mathbf{A}^1}^*(D) \simeq p^*(D')$. Therefore $j_g^* \Phi_{\mathbf{A}^1}^*(D) = g \cdot D \simeq j_g^* p^*(D') = D'$. This implies $h \cdot D \simeq D$ for any $h \in G = \mathrm{ker}_\sigma \cdot \mathrm{Aut}_\nu(\widetilde{X}_\sigma)$

(2) Let $L \subset \widetilde{X}_{\Delta^\sigma}$ denote the complement of the set where the birational morphism $\psi : \widetilde{X}_{\Delta^\sigma} \to \widetilde{X}_\sigma$ is an isomorphism. It is a G^σ-invariant subset of codimension 2. Therefore we get G^σ-equivariant homomorphisms $\mathrm{Cl}(\widetilde{X}_\sigma) \simeq \mathrm{Cl}(\widetilde{X}_\sigma \setminus L) \hookrightarrow \mathrm{Cl}(\widetilde{X}_{\Delta^\sigma})$. Note also that $\mathrm{Cl}(\widetilde{X}_{\Delta^\sigma})$ is generated by $\mathrm{Cl}(\widetilde{X}_\sigma)$ and by the exceptional divisors of ψ which are G^σ-invariant by Lemma 5.3.10. Finally, all elements of $\mathrm{Cl}(\widetilde{X}_{\Delta^\sigma})$ are G^σ-invariant. $\square$

7.6. **Simple definition of a canonical subdivision of a semicomplex.** Recall that for any fan Σ we denote by $\mathrm{Vert}(\Sigma)$ the set of all 1-dimensional rays in Σ.

Proposition 7.6.1. *Let Σ be an oriented Γ-semicomplex. A subdivision Δ of Σ is canonical if for any $\sigma \in \Sigma$*

$$\mathrm{Vert}(\Delta^\sigma) \setminus \mathrm{Vert}(\overline{\sigma}) \subset \mathrm{Stab}(\Sigma).$$

Proof. This is an immediate consequence of Lemma 6.1.2 and the following.

Proposition 7.6.2. *Let σ be a Γ-semicone and Δ^σ be a subdivision of σ. Then the following conditions are equivalent:*

1. $\widetilde{X}_{\Delta^\sigma} \to \widetilde{X}_\sigma$ *is G^σ-equivariant (where $G^\sigma = \mathrm{Aut}(\widetilde{X}_\sigma)^0$).*
2. $\mathrm{Vert}(\Delta^\sigma) \subset \mathrm{Vert}(\overline{\sigma}) \cup \mathrm{Inv}(\sigma)$.
3. $\widehat{X}_{\Delta^\sigma} \to \widehat{X}_\sigma$ *is $\mathrm{Aut}(\widehat{X}_\sigma)^0$-equivariant.*

Before the proof of Propositions 7.6.2 and 7.6.1 we need to show a few lemmas below.

Lemma 7.6.3. *Let σ be a Γ-semicone. If Δ_1 and Δ_2 are subdivisions of σ for which $\widetilde{X}_{\Delta_i} \to \widetilde{X}_\sigma$ is G^σ-equivariant then for the subdivision*

$$\Delta_1 \cdot \Delta_2 := \{\sigma_1 \cap \sigma_2 \mid \sigma_1 \in \Delta_1, \sigma_2 \in \Delta_2\}$$

the morphism $\widetilde{X}_{\Delta_1 \cdot \Delta_2} \to \widetilde{X}_\sigma$ is G^σ-equivariant.

Let I_1 and I_2 be G^σ-invariant on $\widetilde{X}_\sigma$. Let Δ_1 and Δ_2 be subdivisions of σ corresponding to the normalized blow-ups at I_1 and I_2. Then $I_1 \cdot I_2$ corresponds to $\Delta_1 \cdot \Delta_2$.

Proof. It follows from the universal property of the fiber product that $\Delta_1 \cdot \Delta_2$ corresponds to the normalization of an irreducible component in $\widetilde{X}_{\Delta_1} \times_{\widetilde{X}_\sigma} \widetilde{X}_{\Delta_2}$.

The second part of the lemma is an immediate consequence of the relations between ideals and ord-functions: $\mathrm{ord}(I_1 \cdot I_2) = \mathrm{ord}(I_1) + \mathrm{ord}(I_2)$ corresponds to $\Delta_1 \cdot \Delta_2([39])$. $\qquad\square$

Lemma 7.6.4. *Let σ be a Γ-semicone. Let $\tau \subset |\sigma|$ be a cone with all rays $\mathrm{Vert}(\tau)$ in $\mathrm{Vert}(\overline{\sigma}) \cup \mathrm{Inv}(\sigma)$.*

1. *There exists a fan subdivision Δ_τ of σ which contains τ as its face and for which $\widetilde{X}_{\Delta_\tau} \to \widetilde{X}_\tau$ is G^σ-equivariant.*
2. *If all rays of τ which are not in $\mathrm{Inv}(\sigma)$ determine a face ϱ of $|\sigma|$ (and τ) then there exists a G^σ-invariant ideal I such that the normalized blow-up of I corresponds to the subdivision Δ_τ of σ containing τ.*

Proof. Write $\tau = \langle v_1, \dots, v_l \rangle$, where $v_1, \dots, v_k \in \mathrm{Inv}(\sigma)$ and $v_{k+1}, \dots, v_l$ are in $\mathrm{Vert}(\overline{\sigma})$.

(1) Let $\{\tau_i \mid i \in J_0\}$ denote the set of all one-codimensional faces of τ. For any one-codimensional face τ_i of τ, where $i \in J_0$, find an integral functional H_i such that $H_{i|\tau_i} = 0$ and $H_{i|\tau \setminus \tau_i} > 0$. Find functionals H_j, $j \in J_1$ with common zeros exactly on $\mathrm{lin}(\tau)$. Then $\tau = \{x \in \sigma \mid H_i(x) \geq 0, H_j(x) = 0, i \in J_0, j \in J_1\}$.

Let F_0 be an integral functional such that for any $i \in J_0$ and $j \in J_1$, $F_0 + H_i$ and $F_0 - |H_j|$ are strictly greater than zero on $\sigma \setminus \{0\}$. Set $F_0(v_1) = n_1, \dots, F_0(v_l) = n_l$. Let $\nu_1, \dots, \nu_l$ be valuations corresponding to $v_1, \dots, v_l$. Set

$$I := I_{\nu_1, n_1} \cap \dots \cap I_{\nu_l, n_l}.$$

Let $p : \widetilde{X}_{\Delta_0} \to \widetilde{X}_\sigma$ be the composition of the G^σ-equivariant blow-ups $\mathrm{bl}_{\nu_k} \circ \dots \circ \mathrm{bl}_{\nu_1}$ corresponding to G^σ-invariant valuations $\nu_1, \dots, \nu_k$. Then all valuations $\nu_1, \dots, \nu_l$ correspond to Weil divisors $D_1, \dots, D_l$ on $\widetilde{X}_{\Delta_0}$. Set $D = n_1 D_1 + \dots + n_l D_l$. Then

$$p_*(I_D) = \{f \in p_*(\mathcal{O}_{\widetilde{X}_{\Delta_\tau}}) = \mathcal{O}_{\widetilde{X}_\sigma} \mid \nu_i(f) \geq n_i, i = 1, \dots, l\} = I.$$

By Lemma 7.5.2, G^σ acts trivially on $\mathrm{Cl}(\widetilde{X}_{\Delta_0})$; then for any $g \in G$ we have $g_*(D) = D + (f_g) - (h_g)$, where $f_g, h_g \in \mathcal{O}_{\widetilde{X}_\sigma}$. In other words for any $g \in G$,

$$f_g \cdot I_D = h_g \cdot I_{g_*(D)},$$

which implies

$$f_g \cdot I = h_g \cdot g_*(I).$$

This means that the action of G^σ lifts to the blow-up of I, and to its normalization. The normalized blow-up of I corresponds to a G^σ-equivariant subdivision Δ_τ of σ into maximal cones, where the piecewise linear function $\mathrm{ord}(I) = \min\{F \in \sigma^\vee \mid F(v_i) \geq n_i\}$ is linear. We have to show that $\tau \in \Delta_\tau$. Note that $\mathrm{ord}(I)$ is linear on τ since by definition $\mathrm{ord}(I)_{|\tau} = F_{0|\tau}$.

Let $x \notin \tau$. Consider first the case $x \in \mathrm{lin}(\tau)$. There exists a one-codimensional face τ_{i_0}, $i_0 \in J_0$, of τ such that $H_{i_0}(x) < 0$. Then $F_1 := F_0 + H_{i_0} \in \sigma^\vee$ and by definition $F_1 \geq \mathrm{ord}(I)$, since $H_{i_0}(v_i) \geq 0$ and

$F_1(v_i) = F_0(v_i) + H_{i_0}(v_i) \geq n_i$. This implies $\operatorname{ord}(I)(x) \leq F_1(x) < F_0(x)$. Now let $x \notin \operatorname{lin}(\tau)$. Then there is H_{j_0}, $j_0 \in J_1$ such that $H_{j_0}(x) \neq 0$. Set $F_1 := F_0 + H_{j_0}$ if $H_{j_0}(x) < 0$ and $F_1 := F_0 - H_{j_0}$ otherwise. The definition of F_0 shows that F_1 is strictly greater than zero on $\sigma \setminus \{0\}$. We have $F_1(v_i) = F_0(v_i) + H_{j_0}(v_i) = n_i$ and consequently $\operatorname{ord}(I) \leq F_1$. Again $\operatorname{ord}(I)(x) \leq F_1(x) < F_0(x)$. This implies $\operatorname{ord}(I) < F_0$ off $\operatorname{lin}(\tau)$.

(2) The reasoning is very similar. Let J_0 denote the set of all one-codimensional faces τ_i of τ which contain ϱ. For any $i \in J_0$ find an integral functional H_i such that $H_{i|\tau_i} = 0$ and $H_{i|\tau \setminus \tau_i} > 0$. Let H_j, $j \in J_1$, be functionals with common zeros exactly on $\operatorname{lin}(\tau)$. Then $\tau = \{x \in \sigma \mid H_i(x) \geq 0, H_j(x) = 0, i \in J_0, j \in J_1\}$.

Let F_0 be an integral functional such that $F_{0|\tau} = 0$, and for any $i \in I$ and $j \in J$, $F_0 + H_i$, $F_0 - |H_j|$ are strictly greater than zero on $\sigma \setminus \varrho$ for $j \in J_1$. Set $F_0(v_1) = n_1, \dots, F_0(v_k) = n_k$. Then $F_0(v_{k+1}) = F_0(v_l) = 0$. Let $\nu_1, \dots, \nu_k$ be valuations corresponding to $v_1, \dots, v_k$. Set

$$I := I_{\nu_1, n_1} \cap \dots \cap I_{\nu_k, n_k}.$$

By definition I is G^σ-invariant. The normalization of the blow-up of I corresponds to a G^σ-equivariant subdivision Δ_τ of σ into maximal cones, where the piecewise linear function $\operatorname{ord}(I) = \min\{F \in \sigma^\vee \mid F(v_i) \geq n_i\}$ is linear. We have to show that $\tau \in \Delta_\tau$. Note that $\operatorname{ord}(I)$ is linear on τ since by definition $\operatorname{ord}(I)_{|\tau} = F_{0|\tau}$.

The rest of the proof is the same as in (1). $\qquad\square$

Lemma 7.6.5. *Let σ be a Γ-semicone and $\tau \subset |\sigma|$ be a cone all of whose rays are in $\operatorname{Vert}(\overline{\sigma}) \cup \operatorname{Inv}(\sigma)$. Let Δ_τ be a subdivision of σ containing τ as its face and for which $\widetilde{X}_{\Delta_\tau} \to \widetilde{X}_\sigma$ is G^σ-equivariant. Then*

$$S(\tau, \Delta_\tau) := \{\varrho \in \Delta_\tau \mid (\varrho \setminus \tau) \cap \operatorname{Inv}(\sigma) = \emptyset\}$$

is a subfan of Δ_τ for which the open subset $\widetilde{X}_{S(\tau, \Delta_\tau)} = X_{S(\tau, \Delta_\tau)} \times_{X_\sigma} \widetilde{X}_\sigma$ of $\widetilde{X}_{\Delta_\tau}$ is G^σ-invariant.

Proof   It suffices to show that each G^σ-orbit intersecting $\widetilde{X}_{S(\tau, \Delta_\tau)} := X_{S(\tau, \Delta_\tau)} \times_{X_\sigma} \widetilde{X}_\sigma$ is contained in this subscheme. Since $\widetilde{X}_{S(\tau, \Delta_\tau)}$ is an open subset of $\widetilde{X}_{\Delta_\tau}$ any G^σ-orbit contains the generic point of O_ϱ for $\varrho \in S(\tau, \Delta_\tau)$. By Lemma 5.3.15(3), $\operatorname{int}(\varrho)$ intersects $\operatorname{Inv}(\sigma)$. The other toric orbits in the G^σ-orbit correspond to the cones $\varrho' \in \operatorname{Star}(\varrho, \Sigma)$ for which $\varrho' \setminus \varrho = \emptyset$. Thus ϱ is a face of τ and all ϱ' belong to $S(\tau, \Delta_\tau)$. $\qquad\square$

Lemma 7.6.6. *Let σ be a Γ-semicone. Then $S(\tau) := S(\tau, \Delta)$ does not depend upon a subdivision Δ of σ containing τ and all of whose rays are in $\operatorname{Vert}(\overline{\sigma}) \cup \operatorname{Inv}(\sigma)$.*

Proof.   Let Δ_1 and Δ_2 be two subdivisions containing τ and for which all rays are in $\operatorname{Vert}(\overline{\sigma}) \cup \operatorname{Inv}(\sigma)$. By Lemmas 7.6.3 and 7.6.4 we may find subdivisions Δ_ϱ, where $\varrho \in \Delta_1 \cup \Delta_2$ and the common subdivision $\Delta := \prod_{\varrho \in \Delta_1 \cup \Delta_2} \Delta_\varrho$ such that $\widetilde{X}_\Delta \to \widetilde{X}_\sigma$ is G^σ-equivariant. Then $\operatorname{Vert}(\Delta_1 \cdot \Delta_2) \subset \operatorname{Vert}(\Delta) \subset \operatorname{Vert}(\sigma) \cup \operatorname{Inv}(\sigma)$. Hence we can assume that Δ_1 is a subdivision of Δ_2 replacing Δ_1 with $\Delta_1 \cdot \Delta_2$ if necessary. Let $\varrho \in S(\tau, \Delta_2)$. By Lemma 6.5.1(1) applied to Δ_ϱ we can represent ϱ as $\varrho := \varrho' \oplus \langle e_1, \dots, e_k \rangle$, where $\operatorname{int}(\varrho') \cap \operatorname{Inv}(\sigma) \neq \emptyset$, $(\varrho \setminus \varrho') \cap \operatorname{Inv}(\sigma) = \emptyset$ and thus $\varrho' \preceq \tau$. Since all new rays of $\Delta_1|_\varrho$ are in $\operatorname{Inv}(\sigma)$ we find that $\Delta_1|_\varrho = \Delta_1|_{\varrho'} \oplus \langle e_1, \dots, e_k \rangle$. But $\Delta_1|_{\varrho'} = \varrho'$ since $\varrho' \preceq \tau$. Therefore $\Delta_1|_\varrho = \varrho$ and $\varrho \in \Delta_1$, and consequently $\varrho \in S(\tau, \Delta_1)$. $\qquad\square$

Lemma 7.6.7. *Let σ be a Γ-semicone. Let Δ be any subdivision of σ with all "new" rays belonging to $\operatorname{Inv}(\sigma)$. Then for any $\tau \in \Delta$, $\widetilde{X}_{S(\tau)}$ is G^σ-invariant.*

Proof   By Lemma 7.6.4, we find a subdivision Δ_τ of σ. By Lemma 7.6.6, $\widetilde{X}_{S(\tau)} = \widetilde{X}_{S(\tau, \Delta_\tau)}$. By Lemma 7.6.5, the latter scheme is G^σ-invariant. $\qquad\square$

Proof of Proposition 7.6.2. (1) $\Rightarrow$ (2) All exceptional divisors determine G^σ-semiinvariant and therefore G^σ-invariant valuations. The latter correspond to vectors in $\operatorname{Vert}(\Sigma) \subset \operatorname{Vert}(\sigma)$.

(1) $\Leftarrow$ (2) Let Δ be any subdivision of σ containing τ with all "new" rays belonging to $\operatorname{Inv}(\sigma)$. Then by Lemmas 7.6.7, $\widetilde{X}_\Delta$ is the union of open G^σ-invariant neighborhoods $\widetilde{X}_{S(\tau)}$, where $\tau \in \Delta$, and consequently, $\widetilde{X}_\Delta$ is G^σ-invariant.

(2) $\equiv$ (3) By Lemma 7.4.3 vectors defining G^σ-invariant valuations are the same as those defining $\operatorname{Aut}(\widehat{X}_\sigma)^0$-invariant valuations. Then the proof of the equivalence is the same as for (1) $\equiv$ (2). We have to replace $\widetilde{X}_\sigma$ with $\widehat{X}_\sigma$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 7.6.8. *Let σ be a Γ-semicone and G^σ be a connected proalgebraic group acting on $X := \widetilde{X}_\sigma$. Let Δ^σ be a subdivision of σ such that G^K acts on $Y := \widetilde{X}_{\Delta^\sigma}$ and the toric morphism $\psi : Y \to X$ is a proper birational G^K-equivariant morphism. Then there exists a toric ideal I on X such that the normalization of the blow-up of I, $Z \to X$, factors as $Z \to Y \to X$.*

Proof. First subdivide Δ^σ so that all its faces satisfy the condition of Lemma 7.6.4(2). We can make Δ^σ simplicial by applying additional star subdivisions at all its 1-dimensional rays (see [72]). Let τ be a face of Δ^σ which does not satisfy the condition of Lemma 7.6.4(2). Denote by τ' a minimal face of τ with rays in $\text{Vert}(\overline{\sigma})$ and which is not a face of the cone σ. Then there is σ' which is a face of σ and for which $\text{int}(\tau') \subset \text{int}(\sigma')$. Write $\sigma' = \text{sing}^\Gamma(\sigma') \oplus^\Gamma \langle e_1, \dots, e_k \rangle$. Then $\tau' \cap \text{sing}^\Gamma(\sigma')$ is a face of τ' but it is not a face of σ. (Otherwise τ' was a face of σ.) By minimality $\tau' = \tau' \cap \text{sing}^\Gamma(\sigma') \subset \text{sing}^\Gamma(\sigma')$ and $\sigma' = \text{sing}^\Gamma(\sigma')$ is indecomposable and since $\text{int}(\tau') \subset \text{int}(\sigma')$ we can find by Lemma 5.3.15(5) a vector $v \in \text{int}(\tau) \cap \text{Inv}(\sigma)$. By applying star subdivision at $\langle v \rangle$ to Σ we eliminate the cone τ (and some others cones not satisfying the condition of Lemma 7.6.4(2)). After a finite number of steps we arrive at a subdivision Δ^σ with all faces satisfying the condition of Lemma 7.6.4(2).

Then by Lemma 7.6.4(2), for any $\tau \in \Delta$, we construct the ideal I_τ such that the normalization of the blow-up of I_τ corresponds to the subdivision Δ^σ_τ of σ containing τ. Then by Lemma 7.6.3 the G^σ-invariant ideal $I := \prod_{I_\tau \in \Sigma} I_\tau$ correponds to the subdivision $\Delta_I := \prod_{\tau \in \Sigma} \Delta^\sigma_\tau$ of σ. $\qquad\square$

Lemma 7.6.9. *Let σ be a face of a fan Σ. Let Δ_1 be a subdivision of σ such that $\text{Vert}(\Delta_1) = \text{Vert}(\sigma)$. Then there is a subdivision Δ_2 of Σ such that $\Delta_2|\sigma = \Delta_1$ and $\text{Vert}(\Delta_2) = \text{Vert}(\Sigma)$.*

Proof. Set $\{v_1, \dots, v_k\} = \text{Vert}(\Sigma) \setminus \text{Vert}(\sigma)$, $\Delta_0 := \langle v_1 \rangle \cdots \langle v_k \rangle \cdot \Sigma$. Then $\sigma \in \Delta_0$. We shall construct a subdivision Δ_2 for any cone in Δ_0. If $\tau \in \Delta_0$ does not intersect σ then we put $\Delta_2|\tau = \tau$. If $\tau \in \Delta_0$ intersects σ along a common face σ_0 then all rays $\text{Vert}(\tau \setminus \sigma_0)$ are centers of star subdivisions and therefore are linearly independent of the other rays of σ and in particular of $\text{lin}(\sigma_0)$ (After a star subdivision at $\langle v \rangle$, v becomes linearly independent of all other rays of cones containing v. Hence the subdivision $\Delta_2|\sigma_0$ extends to a unique subdivision of τ with no new rays, $\Delta_2|\tau := \{\delta + \sum_{\varrho \in (\text{Vert}(\tau) \setminus \text{Vert}(\sigma_0))} \varrho \mid \delta \in \Delta|\sigma_0\}$. $\qquad\square$

Lemma 7.6.10. *Let $\tau \subset \sigma$ be a cone such that $\text{Vert}(\tau) \subset \text{Vert}(\sigma)$. Then there is a proper subdivision Δ of σ containing τ and such that $\text{Vert}(\Delta) = \text{Vert}(\sigma)$.*

Proof. By Lemma 7.6.9 it suffices to prove the lemma for the situation when $\text{Vert}(\sigma) \setminus \text{Vert}(\tau)$ consists of one ray. Then for any τ we find σ' containing τ and for which $\text{card}(\text{Vert}(\sigma) \setminus \text{Vert}(\tau)) = 1$. By induction we can find a subdivision of σ' and then by Lemma 7.6.9 extend it to a subdivision of σ.

Let $\tau = \langle v_1, \dots, v_k \rangle$ and $\sigma = \langle v_1, \dots, v_k, v_{k+1} \rangle$. If τ is simplicial and τ is not a face of σ then there is a unique linear relation between $v_1, \dots, v_k, v_{k+1}$. Without loss of generality we can assume that it is $a_r v_r + a_{r+1} v_{r+1} + \dots + a_l v_l = a_{l+1} v_{l+1} + \dots + a_{k+1} v_{k+1}$, where $r \geq 1$ and all coefficients are positive. Then there are exactly two simplicial subdivisions of σ with no "new" rays and with maximal cones $\langle v_1, \dots, \check{v}_i, \dots, v_k, v_{k+1} \rangle$, where $i = r, \dots l$ and $i = l + 1, \dots, k + 1$ (see [72]). The second subdivision contains τ. If $\tau = \langle v_1, \dots, v_k \rangle$ is not simplicial then for the cone $\tau^\vee$ we find a simplicial cone $(\tau_0^\vee, M_{\tau_0})$ and a linear epimorphism $\phi^\vee : (\tau_0^\vee, M_{\tau_0} \to (\tau^\vee, M_\tau)$ mapping rays of $\tau_0^\vee$ to rays of $\tau_0^\vee$. Moreover let $H^\vee \subset M_{\tau_0}$ denote the vector subspace which is the kernel of $\phi^\vee$. Then $\tau^\vee \simeq (\tau_0^\vee + H^\vee)/H^\vee$. The dual morphism $\phi : (\tau, N_\tau) \to (\tau_0, N_{\tau_0})$ is a monomorphism which maps τ isomorphically to $\tau_0 \cap H$, where $H = \{v \in N_{\tau_0} \mid v_{|H^\vee} = 0\}$. Then also $v_{k+1} \in H \setminus \tau_0$. Note that $v_{k+1} \notin -\tau_0$ since otherwise $v_{k+1} \in (-\tau_0) \cap H = -\tau$. Thus we can apply the previous case to $\sigma_0 := \langle v_{k+1} \rangle + \tau_0 \supset \tau_0$ and obtain a subdivision Δ_0 of σ_0 containing τ_0 with no "new " rays. Intersecting Δ_0 with H defines a subdivision Δ of σ containing τ. All rays in Σ except $v_{k+1} \in H$ are obtained by intersecting some faces of τ_0 with H, hence belong to $\text{Vert}(\tau)$. $\qquad\square$

8. ORIENTABILITY OF STRATIFIED TOROIDAL VARIETIES AND RESOLUTION OF SINGULARITIES

8.1. **Orientation group of an affine toric variety.**

Definition 8.1.1. Let (X, S) be a stratified noetherian scheme over K and $x \in X$ be a K-rational point fixed under the Γ-action. By the *orientation group* of (X, S) at $x \in X$ we mean $\Theta^\Gamma(X, S, x) := \text{Aut}(\widehat{X}_{X,x})/\text{Aut}(\widehat{X}_{X,x})^0$.

Lemma 8.1.2. *Let σ be a Γ-semicone. Let $\widehat{\text{Inv}}(\sigma) = \text{Inv}(\sigma)$ be the set of the vectors $v \in \sigma$ corresponding to $\text{Aut}(\widehat{X}_\sigma)^0$-invariant valuations $\text{val}(v)$ on $\widehat{X}_\sigma$ (see Lemmma 7.4.3). Let $\text{Aut}(\sigma)_{\text{Inv}}$ be the group of all elements g of $\text{Aut}(\sigma)$ satisfying the following conditions:*

1. $g_{|\mathrm{Inv}(\sigma)} = \mathrm{id}_{|\mathrm{Inv}(\sigma)}$.
2. *For any subdivision Δ of σ such that $\mathrm{Vert}(\Delta) = \mathrm{Vert}(\overline{\sigma})$, g lifts to an automorphism of Δ.*

   Then $\mathrm{Aut}(\sigma)_{\mathrm{Inv}} = Aut(\sigma)^0$ and $\Theta^\Gamma(X_\sigma, S, O_\sigma) = \mathrm{Aut}(\sigma)/\mathrm{Aut}(\sigma)_{\mathrm{Inv}}$.

Proof. By Lemma 7.3.2 there is an epimorphism $\mathrm{Aut}(\sigma) \longrightarrow \Theta^\Gamma(X_\sigma, S, O_\sigma)$ with kernel $\mathrm{Aut}(\sigma)^0$. By definition all $g \in \mathrm{Aut}(\sigma)^0$ preserve $\mathrm{Inv}(\sigma)$. By Proposition 7.6.1 the subdivisions Δ satisfying condition (2) of the lemma determine $\mathrm{Aut}(\widehat{X}_\sigma)^0$-equivariant morphisms $\widehat{X}_\Delta \to \widehat{X}_\sigma$ which commute with morphisms defined by $g \in \mathrm{Aut}(\sigma)^0$. Thus automorphisms $g \in \mathrm{Aut}(\sigma)^0$ lift to automorphisms of Δ. Therefore $\mathrm{Aut}(\sigma)^0 \subseteq \mathrm{Aut}(\sigma)_{\mathrm{Inv}}$. We need to prove that $\mathrm{Aut}(\sigma)^0 \supseteq \mathrm{Aut}(\sigma)_{\mathrm{Inv}}$.

Let $v_1, \ldots, v_k$ denote the generators of σ. Let σ_0 denote the regular cone spanned by the standard basis $e_1, \ldots, e_k$ of some k-dimensional lattice. Let $\pi : \sigma_0 \to \sigma$ be the projection defined by $\pi(e_i) = v_i$. Each automorphism of σ lifts to a unique automorphism of σ_0. This defines a group homomorphism $\pi^* : \mathrm{Aut}(\sigma)_{\mathrm{Inv}} \to \mathrm{Aut}(\sigma_0)$. Let $\mathrm{Aut}(\sigma_0)_{\mathrm{Inv}}$ denote the group of automorphisms of the cone σ_0 preserving $\pi^{-1}(\mathrm{Inv}(\sigma))$.

Lemma 8.1.3. $\mathrm{Aut}(\sigma_0)_{\mathrm{Inv}} \simeq \mathrm{Aut}(\sigma)_{\mathrm{Inv}}$.

Proof. Any automorphism $g \in \mathrm{Aut}(\sigma)_{\mathrm{Inv}}$ preserves all vectors from $\mathrm{Inv}(\sigma)$ and consequently it preserves the vectors from $\overline{\mathrm{Inv}(\sigma)} := \mathrm{lin}(\mathrm{Inv}(\sigma)) \cap \sigma = \mathrm{lin}(\overline{\mathrm{Inv}(\sigma)}) \cap \sigma$. Set $\overline{\mathrm{Inv}(\sigma_0)} := \pi^{-1}(\overline{\mathrm{Inv}(\sigma)}) = \langle w_1, \ldots, w_m \rangle$. Then, by definition $\overline{\mathrm{Inv}(\sigma_0)} = \mathrm{lin}(\overline{\mathrm{Inv}(\sigma_0)}) \cap \sigma_0$.

Let $g' := \pi^*(g)$. We shall show that g' preserves $\overline{\mathrm{Inv}(\sigma_0)}$.

Let $w_i \in \mathrm{int}(\tau_i)$, where $\tau_i = \langle e_{ij_1}, \ldots, e_{ij_l} \rangle \preceq \sigma_0$. Then , by definition, w_i defines a unique ray of the cone $\tau_i \cap \overline{\mathrm{Inv}(\sigma_0)} = \tau_i \cap \mathrm{lin}(\overline{\mathrm{Inv}(\sigma_0)})$. Then $\ker(\pi) \subset \mathrm{lin}(\overline{\mathrm{Inv}(\sigma_0)})$ and $\ker(\pi) \cap \mathrm{lin}(\tau_i) = \ker(\pi) \cap \mathrm{lin}(\overline{\mathrm{Inv}(\sigma_0)}) \cap \mathrm{lin}(\tau_i) = \ker(\pi) \cap \mathrm{lin}\{w_i\} = 0$. Consequently π maps τ_i isomorphically onto a cone $\pi(\tau_i) = \langle v_{ij_1}, \ldots, v_{ij_l} \rangle$ containing a unique ray $\pi(w_i) \in \overline{\mathrm{Inv}(\sigma)}$. By Lemma 7.6.10 such a cone is a face of some subdivision Δ of σ satisfying condition (2) of the lemma. Thus $\pi(\tau_i)$ is a unique face of Δ containing $\pi(w_i)$ in its interior. Therefore it is preserved by g. Consequently, g' preserves τ_i and w_i, and hence all points from $\overline{\mathrm{Inv}(\sigma_0)}$.

Now, let g' be an automorphism of σ_0 preserving $\overline{\mathrm{Inv}(\sigma_0)}$. Then g' defines a permutation g of $v_1, \ldots, v_k$.

We will show that such a permutation preserves linear relations between $v_1, \ldots, v_k$.

Let $p := a_{i_1} v_{i_1} + \ldots + a_{i_r} v_{i_r} = a_{i_{r+1}} v_{i_{r+1}} + \ldots + a_{i_s} v_{i_s}$ be a minimal linear relation, i.e. $s - 1 = \dim\{v_{i_1}, \ldots, v_{i_s}\}$. Then both cones $\sigma_1 := \langle v_{i_1}, \ldots, v_{i_r} \rangle$ and $\sigma_2 := \langle v_{i_{r+1}}, \ldots, v_{i_\sigma} \rangle$ are simplicial and by Lemma 7.6.10 they are faces of two subdivisions Δ_1 and Δ_2 of σ satisfying the condition 2 of the lemma. Then $\langle p \rangle \in \mathrm{Vert}(\Delta_1 \cdot \Delta_2)$ and by Lemma 7.6.3 and Proposition 7.6.2, $p \in \mathrm{Inv}(\sigma)$. Consequently the point $p' := a_{i_1} e_{i_1} + \ldots + a_{i_r} e_{i_r} \in \overline{\mathrm{Inv}(\sigma_0)}$. This implies $p' = g'(p') = a_{i_1} g'(e_{i_1}) + \ldots + a_{i_r} g'(e_{i_r})$ and $p = a_{i_1} v_{i_1} + \ldots + a_{i_r} v_{i_r} = a_{i_1} g(v_{i_1}) + \ldots + a_{i_r} g(v_{i_r})$. Analogously $p = a_{i_{r+1}} g(v_{i_{r+1}}) + \ldots + a_{i_s} g(v_{i_s})$. Finally, $a_{i_1} g(v_{i_1}) + \ldots + a_{i_r} g(v_{i_r}) = a_{i_{r+1}} g(v_{i_{r+1}}) + \ldots + a_{i_s} g(v_{i_s})$ and g defines a linear automorphism preserving vectors from $\overline{\mathrm{Inv}(\sigma)}$.

We need to show that g satisfies condition (2). Let Δ be a subdivision satisfying (2). If τ is any face of Δ whose relative interior intersects $\mathrm{Inv}(\sigma) \subset \overline{\mathrm{Inv}(\sigma)}$ then it lifts to a face τ' of σ_0 whose relative interior intersect $\overline{\mathrm{Inv}(\sigma_0)}$. Thus σ' is preserved by g' and σ is preserved by g. Therefore all faces of Δ whose relative interior intersect $\mathrm{Inv}(\sigma)$ are preserved. If the relative interior of $\tau \in \Delta$ does not intersect $\mathrm{Inv}(\sigma)$ then $\tau \in S(\sigma') \subset \Delta$, where $\sigma' \in \Delta$, is the maximal face of τ whose relative interior intersects $\mathrm{Inv}(\sigma)$ (see Lemma 7.6.5). By the above, σ' is preserved by g. Thus by 7.6.6 $g(\tau) \in S(\sigma', g(\Delta)) = S(\sigma', \Delta) = S(\sigma')$. Consequently, all faces of Δ are mapped to faces of Δ and finally g defines an automorphism of Δ. Lemma 8.1.3 is proven. $\square$

Write $X_\sigma = X_{\sigma_0}//T_0$ for the torus T_0 corresponding to the kernel of π. Also $\widehat{X}_\sigma = \widehat{X}_{\sigma_0}//T_0$ and $K[\widehat{X}_\sigma] = K[\widehat{X}_{\sigma_0}]^{T_0}$.

Let $\mathrm{Aut}(\widehat{X}_{\sigma_0})_{\mathrm{Inv}}$ denote the group of all T_0-equivariant automorphisms g in $\mathrm{Aut}(\widehat{X}_{\sigma_0})$ which preserve invariant valuations, i.e $g_*(\mathrm{val}(v)) = \mathrm{val}(v)$ for any $v \in \overline{\mathrm{Inv}(\sigma_0)}$. By definition each automorphism $g \in \mathrm{Aut}(\widehat{X}_{\sigma_0})_{\mathrm{Inv}}$ defines an automorphism of $K[\widehat{X}_{\sigma_0}]^{T_0}$. This gives the homomorphism $p : \mathrm{Aut}(\widehat{X}_{\sigma_0})_{\mathrm{Inv}} \longrightarrow \mathrm{Aut}(\widehat{X}_\sigma)$. It now suffices to prove the following Lemma:

Lemma 8.1.4. $\mathrm{Aut}(\widehat{X}_{\sigma_0})_{\mathrm{Inv}}$ *is connected.*

Consequently, $\mathrm{Aut}(\sigma)_{\mathrm{Inv}} \subset p(\mathrm{Aut}(\widehat{X}_{\sigma_0})_{\mathrm{Inv}}) \subset \mathrm{Aut}(\widehat{X}_\sigma)^0$ and $\mathrm{Aut}(\sigma)_{\mathrm{Inv}} \subset \mathrm{Aut}(\sigma)^0$. $\square$

Proof of Lemma 8.1.4. The same reasoning as in Lemma 4.9.9. Let $x_1, \ldots, x_k$ be the standard coordinates on $X_{\sigma_0} = \mathbf{A}^k$. Any automorphism $g \in \mathrm{Aut}(\widehat{X}_{\sigma_0})_{\mathrm{Inv}}$ is given by Γ-semiinvariant functions $g^*(x_1), \ldots, g^*(x_k)$ with the corresponding Γ-weights and such that $\mathrm{val}(v)(g^*(x_i)) = \mathrm{val}(v)((x_i))$, where $v \in \overline{\mathrm{Inv}}(\sigma_0) \cap N_{\sigma_0}$. There is a birational map $\alpha : \mathbf{A}^1 \to G$ defined by

$$\alpha(z) := (x_1, \ldots, x_k) \mapsto ((1-z)g^*(x_1) + zx_1, \ldots, (1-z)g^*(x_k) + zx_k).$$

$\alpha(z)$ defines a Γ-equivarinat automorphism of $\widehat{X}_{\sigma_0}$ for all z in the open subset U of $\mathbf{A}^1$, where the linear parts of coordinates of $\alpha(z)$ are linearly independent. Note that for any integral vector $v \in \overline{\mathrm{Inv}}(\sigma_0)$ and $z \in U$, $\mathrm{val}(v)((1-z)g^*(x_i) + zx_i) \geq \mathrm{val}(v)(x_i)$. By Lemma 5.1.4, $\alpha(z)$ preserves $\mathrm{val}(v)$. □

Example 8.1.5. Let $X_\sigma \subset \mathbf{A}^4$ be a toric variety described by $x_1x_2 = x_3x_4$. Then $\sigma = \langle v_1, v_2, v_3, v_4 \rangle \subset N_\sigma^{\mathbf{Q}}$ is a cone over a square, where $N_\sigma^{\mathbf{Q}} := \{(x_1, x_2, x_3, x_4) \mid x_1 + x_2 - x_3 - x_4\} \subset \mathbf{Q}^4$, $v_1 = (1, 0, 0, 1)$, $v_2 = (0, 1, 1, 0)$, $v_3 = (1, 0, 1, 0)$ $v_4 = (0, 1, 0, 1)$. Then $\mathrm{Aut}(\sigma) = \mathbf{D}_8$ consists of all isometries of σ. $\mathrm{Aut}(\sigma)_{\mathrm{Inv}}$ consists of all isometries preserving diagonals: the reflections with respect to the diagonals and rotation through π. Consequently

$$\Theta(X_\sigma, S, O_\sigma) = \mathrm{Aut}(\sigma)/\mathrm{Aut}(\sigma)_{\mathrm{Inv}} \simeq \mathbf{Z}_2.$$

8.2. Stratified toroidal varieties are orientible.

Lemma 8.2.1. *Let $\phi_i : X \to X_\sigma$ for $i = 1, 2$ be Γ-smooth morphisms of a Γ-stratified toroidal scheme (X, S) to a Γ-stratified toric variety (X_σ, S_σ) such that strata of S are precisely the inverse images of strata of S_σ. Denote by $\Delta_1, \ldots, \Delta_l$ all subdivisions of σ for which $\mathrm{Vert}(\Delta_i) = \mathrm{Vert}(\overline{\sigma})$. Let $v_1, \ldots, v_r$ be stable vectors such that $\mathrm{lin}(v_1, \ldots, v_r) \supseteq \mathrm{Inv}(\sigma)$. Denote by $\Delta_{l+1}, \ldots, \Delta_{l+r}$ the star subdivisions of σ at $\langle v_1 \rangle, \ldots, \langle v_r \rangle$. The following conditions are equivalent:*

1. *ϕ_1 and ϕ_2 determine the same orientation at a K-rational point $x \in s = \phi_i^{-1}(\mathrm{strat})\sigma))$.*
2. *There is an open neighborhood U of x such that $U_{j1} := U \times_{X_\sigma} X_{\Delta_j}$, defined via ϕ_1, and $U_{j2} := U \times_{X_\sigma} X_{\Delta_j}$, defined via ϕ_2, are isomorphic over U for any $j = 1, \ldots, l + r$.*

Proof. ($\Rightarrow$) Suppose ϕ_1 and ϕ_2 determine the same orientation at x. By Proposition 7.6.1 the morphism $\widetilde{X}_{\Delta_j} \to \widetilde{X}_\sigma$ is G^σ-equivariant. Therefore by Lemma 4.13.2 for any subdivision Δ_j of σ there is an open neighborhood U_j of x such that $U'_{j1} := U_j \times_{X_\sigma} X_{\Delta_j}$, defined via ϕ_1, and $U'_{j2} := U_j \times_{X_\sigma} X_{\Delta_j}$, defined via ϕ_2, are isomorphic over U_j. It suffices to put $U = \bigcap U_j$.

($\Leftarrow$) Suppose ϕ_1 and ϕ_2 do not determine the same orientation at x. Then by Lemma 8.1.2 we find an automorphism ϕ of σ such that for the corresponding automorphism $\overline{\phi} \in \mathrm{Aut}(X_\sigma)$, ϕ_1 and $\overline{\phi}\phi_2$ determine the same orientation at x. By the above there is an open $U \ni x$ for which U_{j,ϕ_1} and $U_{j,\overline{\phi}\phi_2}$ are isomorphic. But $\overline{\phi}$ does not preserve orientation so by 8.1.2, ϕ does not satisfy one of the conditions (1)-(2) of the lemma and does not lift to an automorphism of some $\Delta_j \neq \phi(\Delta_j)$. Consequently, $U_{j,\phi\phi_2} \simeq U_{j,\phi_1} \simeq U \times_{X_\sigma} X_{\Delta_j}$ is not isomorphic to $U_{j,\phi_2} \simeq U \times_{X_\sigma} X_{\phi(\Delta_j)}$ for any open neighborhood U of x. □

Lemma 8.2.2. *Let (X_Δ, S_Ω) be a Γ-stratified toric variety corresponding to an embedded semifan $\Omega \subset \Delta$. Let $\phi_i : X \to X_\Delta$ for $i = 1, 2$ be Γ-smooth morphisms of a Γ-stratified toroidal scheme (X, S) to a Γ-stratified toric variety (X_Δ, S_Ω) such that the strata of S are precisely the inverse images of strata of S_Ω.*

For a cone $\omega \in \Omega$ denote by $\omega(\Delta)$ the subset $\{\tau \in \Delta \mid \omega(\tau) = \omega\}$ of Δ. Let $\Delta_1, \ldots, \Delta_l$ be all subdivisions of ω satisfying condition (2) of Lemma 8.1.2. Let $\Delta_{l+1}, \ldots, \Delta_{l+r}$ denote the star subdivisions of ω at stable vectors $\langle v_1 \rangle, \ldots, \langle v_r \rangle$ for which $\mathrm{lin}(v_1, \ldots, v_r) \supseteq \mathrm{Inv}(\sigma)$. Denote by $\Delta_j(\Delta)$ the subdivisions of $\omega(\Delta)$ induced by Δ_j, i.e. for any $\delta = \omega(\delta) \oplus r(\delta) \in \omega(\Delta)$, $\Delta_j(\Delta)|\delta = \Delta_j \oplus r(\delta)$. Let $x \in \phi_i^{-1}(\mathrm{strat}(\omega))$ be a K-rational point. The following conditions are equivalent:

1. *$\phi_1^*(\mathcal{U}(\Delta, \Omega))$ and $\phi_2^*(\mathcal{U}(\Delta, \Omega))$ are compatible at x.*
2. *There is an open neighborhood U of x such that $U_{j1} := U \times_{X_{\omega(\Delta)}} X_{\Delta_j(\Delta)}$, defined via ϕ_1, and $U_{j2} : U \times_{X_{\omega(\Delta)}} X_{\Delta_j(\Delta)}$, defined via ϕ_2, are isomorphic over U for any $j = 1, \ldots, l + r$*

Proof. Let $\phi_i(x) \in \delta_i$, where $\delta_i \in \Delta$ and $i = 1, 2$. Denote by U_i the inverse image $\phi_i^{-1}(X_{\delta_i})$. Since $x \in \phi_i^{-1}(\mathrm{strat}(\sigma))$, we have $\omega(\delta_i) = \omega$. The atlases $\phi_i^*(\mathcal{U}^{\mathrm{can}}(X_\Delta, S_\Delta))$ are compatible at x if the morphisms $\pi_\sigma^{\sigma_i}\phi_i : U_i \to X_\omega$ determine the same orientation at x. This is equivalent by Lemma 8.2.1 to the fact that $U \times_{X_\sigma} X_{\Delta_j}$, defined via $\pi_\sigma^{\sigma_i}\phi_i$, are isomorphic for some U. The latter schemes are isomorphic to $U \times_{X_{\sigma_i}} X_{\Delta_j(\Delta)|\delta_i} = U \times_{X_{\omega(\Delta)}} X_{\Delta_j(\Delta)}$. □

Lemma 8.2.3. *Let $\phi_i : (X, S) \to (X_\Delta, S_\Omega)$ for $i = 1, 2$ be Γ-smooth morphisms of a Γ-stratified toroidal scheme to a Γ-stratified toric variety (X_Δ, S_Ω) such that the strata of S are precisely the inverse images of strata of S_Ω. Assume that a stratum $s \in S$ is a variety over K and $\phi_1^*(\mathcal{U}(\Delta, \Omega))$ and $\phi_2^*(\mathcal{U}(\Delta, \Omega))$ are compatible at some point $x \in s$. Then they are compatible along s.*

Proof. By Lemma 8.2.2, $\phi_1^*(\mathcal{U}(\Delta, \Omega))$ and $\phi_2^*(\mathcal{U}(\Delta, \Omega))$ are compatible at any $x \in s$ iff they are compatible at all points of $s \cap U_x$, where U_x is an open neighborhood of x. Since s is a variety the sets $s \cap U_x$ and $s \cap U_y$ intersect for any $x, y \in s$.   □

Lemma 8.2.4. *Let $\phi_1, \phi_2 : X \to X_\sigma$ be two Γ-smooth morphisms of a Γ-stratified toroidal variety (X, S) into the Γ-stratified toric variety X_σ, such that the strata of S are precisely the inverse images of strata of S_σ. Assume that ϕ_1, ϕ_2 determine the same orientation at a point $x \in \phi_i^{-1}(\mathrm{strat}(\sigma))$. Then $\phi_i^*(\mathcal{U}^{\mathrm{can}}(X_\sigma, S_\sigma))$ and $\phi_2^*(\mathcal{U}^{\mathrm{can}}(X_\sigma, S_\sigma))$ are compatible on X.*

Proof. Let $\Delta_1, \ldots, \Delta_l, \ldots, \Delta_{l+r}$ be subdivisions of $\tau \leq \sigma$ as in Lemma 8.2.1. By Lemma 7.6.9, the subdivisions $\Delta_1, \ldots, \Delta_l$ can be extended to subdivisions $\Delta_1(\sigma), \ldots, \Delta_l(\sigma)$ of σ for which $\mathrm{Vert}(\Delta_i) = \mathrm{Vert}(\overline{\sigma})$. By Lemma 7.4.4 $\mathrm{Inv}(\tau) \subset \mathrm{Inv}(\sigma)$ and we can extend the star subdivisions $\Delta_{l+1}, \ldots, \Delta_{l+r}$ to the star subdivisions $\Delta_{l+1}(\sigma), \ldots, \Delta_{l+r}(\sigma)$ of σ at $v_1, \ldots, v_r \in \mathrm{Inv}(\tau) \subset \mathrm{Inv}(\sigma)$. By Lemma 4.13.2 there is an open neighborhood U of x for which $U_{j,\phi_i} = U \times_{X_\sigma} X_{\Delta_j}$ definded for ϕ_1 and ϕ_2 are pairwise isomorphic for any j. Let $\tau(\sigma) = \{\varrho \preceq \sigma \mid \omega(\varrho) = \tau\}$. Then there is an open subset $U' = \phi_1^{-1}(X_{\tau(\sigma)}) \cap \phi_2^{-1}(X_{\tau(\sigma)}) \cap U$ intersecting $\phi_i^{-1}(\mathrm{strat}(\tau))$ such that $U' \times_{X_\sigma} X_{\Delta_j} = U' \times_{X_{\tau(\sigma)}} X_{\Delta_j(\sigma)|\tau(\sigma)}$ defined by the restrictions of ϕ_i are isomorphic. This implies by Lemma 8.2.2 that $\phi_1^*(\mathcal{U}^{\mathrm{can}}(X_\sigma, S_\sigma))$ and $\phi_2^*(\mathcal{U}^{\mathrm{can}}(X_\sigma, S_\sigma))$ are compatible at $x \in \mathrm{strat}(\tau) \cap U'$ and consequently by Lemma 8.2.3 they are compatible at all the points of $\phi_i^{-1}(\mathrm{strat}(\tau))$.   □

Lemma 8.2.5. *The Γ-semicomplex Σ associated to an oriented Γ-stratified toroidal variety (X, S) is oriented.*

Proof. For any $\sigma \leq \tau$, the inclusion morphism $\iota_\sigma^\tau : \sigma \to \tau$ identifies σ with a face of τ. Let $\overline{\pi}_\tau^\sigma : N_\tau \to N_\sigma$ denote a projection such that $\overline{\pi}_\tau^\sigma \circ \iota_\sigma^\tau = \mathrm{id}_{|\sigma}$. Let $\pi_\tau^\sigma : X_\sigma \to X_\tau$ denote the induced toric morphism. Consider faces $\sigma \leq \tau \leq \varrho$ of the semicomplex Σ. Let $\phi_\sigma : U_\sigma \to X_\sigma, \phi_\tau : U_\tau \to X_\tau, \phi_\varrho : U_\varrho \to X_\varrho$ denote any charts on (X, S) corresponding to the cones σ, τ and ϱ. Since $\tau \leq \varrho$, the open subset U_ϱ intersects τ. For a generic point x_τ of $\mathrm{strat}(\tau) \cap U_\varrho$, the morphism ϕ_ϱ sends x_τ to the toric orbit $O_\tau \subset X_\varrho$. Since (X, S) is oriented we find that $\pi_\varrho^\tau \phi_\varrho : U_\varrho^\tau \to X_\tau$ and $\phi_\tau : U_\tau \to X_\tau$ determine the same orientation at x_τ. Analogously $\pi_\varrho^\sigma \phi_\varrho, \phi_\sigma$ and $\pi_\tau^\sigma \phi_\tau$ determine the same orientation for a generic point x_σ of $\mathrm{strat}(\sigma)$. Applying Lemma 8.2.4 to $\pi_\varrho^\tau \phi_\varrho$ and ϕ_τ we see that $\pi_\tau^\sigma \pi_\varrho^\tau \phi_\varrho$ and $\pi_\tau^\sigma \phi_\tau$ determine the same orientation for a generic point of $\mathrm{strat}(\sigma)$. Finally $\pi_\tau^\sigma \pi_\varrho^\tau \phi_\varrho$ and $\pi_\varrho^\sigma \phi_\varrho$ determine the same orientation and thus so do $\pi_\tau^\sigma \pi_\varrho^\tau$ and π_ϱ^σ. Write $\iota_\tau^\varrho \iota_\sigma^\tau = \iota_\sigma^\varrho \alpha_\sigma$, where $\alpha_\sigma \in \mathrm{Aut}(\sigma)$. Then $\alpha_\sigma \pi_\tau^\sigma \pi_\varrho^\tau \iota_\sigma^\varrho = \alpha_\sigma \pi_\tau^\sigma \pi_\varrho^\tau \iota_\sigma^\varrho \iota_\sigma^\tau \alpha_\sigma^{-1} = \mathrm{id}_{|\sigma}$. By Lemma 4.9.14 the toric morphisms induced by the projections $\alpha_\sigma \pi_\tau^\sigma \pi_\varrho^\tau$ and π_ϱ^σ determine the same orientation. Thus the morphisms induced by $\alpha_\sigma \pi_\tau^\sigma \pi_\varrho^\tau$ and $\pi_\tau^\sigma \pi_\varrho^\tau$ determine the same orientation, which gives $\alpha_\sigma \in \mathrm{Aut}(\sigma)^0$.   □

Lemma 8.2.6. *On any Γ-stratified toroidal variety (X, S) there exists an atlas $\mathcal{U}$ such that $(X, S, \mathcal{U})$ is oriented.*

Proof. We shall improve the charts from $\mathcal{U}$ and inclusion maps in the associated Γ-semicomplex Σ. For any $\sigma \in \Sigma$ fix a chart $\phi_{\sigma, a_\sigma} : U_{\sigma, a_\sigma} \to X_\sigma$. By Lemma 7.3.2 for any chart $\phi_{\sigma, a} : U_{\sigma, a} \to X_\sigma$ we can find $\psi_{\sigma, a}$ of X_σ) induced by an automorphism of σ such that ϕ_{σ, a_σ} and

$$\overline{\phi}_{\sigma, a} := \psi_{\sigma, a} \circ \phi_{\sigma, a}$$

determine the same orientation for some points of $\mathrm{strat}(\sigma) \cap U_{\sigma, a_\sigma} \cap U_{\sigma, a}$. By Lemma 8.2.3 the above morphisms determine the same orientation for all points from $\mathrm{strat}(\sigma) \cap U_{\sigma, a_\sigma} \cap U_{\sigma, a}$.

Let $\sigma \leq \tau$. Denote by $\sigma' \preceq \tau$ the face cone in N_τ such that $\underline{\sigma'} = \sigma$. By Lemma 7.3.2 find $\alpha_\sigma \in \mathrm{Aut}(\sigma)$ such that for the inclusion map $\overline{\iota_{\sigma'}^\tau} := \iota_{\sigma'}^\tau \circ \alpha_\sigma$ any two charts $\phi_{\sigma, a_\sigma} : U_{\sigma, a_\sigma} \to X_\sigma, \pi_{\sigma'}^\sigma \phi_{\tau, a_\tau}^{\sigma'} : U_{\tau, a_\tau}^{\sigma'} \to X_\sigma$ define the same orientation at some point of $x \in \mathrm{strat}(\sigma) \cap U_{\sigma, a_\sigma} \cap U_{\tau, a_\tau}^{\sigma'}$. By Lemma 8.2.3 they determine the same orientation at all points of $\mathrm{strat}(\sigma) \cap U_{\sigma, a_\sigma} \cap U_{\tau, a_\tau}^{\sigma'}$. We need to verify that for any two charts $\phi_{\sigma, a}$ and $\phi_{\tau, b}$, where $\sigma \leq \tau$, and any $\tau' \preceq \tau$ such that $\omega(\tau') = \sigma$ the morphisms $\phi_{\sigma, a}$ and $\phi_{\tau, b}^{\tau'}$ determine the same orientation for all $x \in \mathrm{strat}(\sigma) \cap U_{\sigma, a} \cap U_{\tau, b}^{\tau'}$. By Lemma 8.2.3 it suffices to show that the above morphisms determine the same orientation for some $x \in \mathrm{strat}(\sigma) \cap U_{\sigma, a} \cap U_{\sigma, a_\sigma} \cap U_{\tau, b}^{\tau'} \cap U_{\tau, a_\tau}^\sigma$. By the above $\phi_{\sigma, a}$ and ϕ_{σ, a_σ} determine

the same orientation at all $x \in \operatorname{strat}(\sigma) \cap U_{\sigma,a} \cap U_{\sigma,a_\sigma}$. Also $\phi_{\tau,b}$ and ϕ_{τ,a_τ} determine the same orientation at all $x \in \operatorname{strat}(\tau) \cap U_{\tau,b} \cap U_{\tau,a_\tau}$. By Lemma 8.2.4, $\phi_{\tau,b}^*(\mathcal{U}^{\operatorname{can}}(X_\tau, S_\tau))$ and $\phi_{\tau,a_\tau}^*(\mathcal{U}^{\operatorname{can}}(X_\tau, S_\tau))$ are compatible on $U_{\tau,b} \cap U_{\tau,a_\tau}$. Hence $\phi_{\tau,b}^{\tau'}$ and $\phi_{\tau,a_\tau}^{\sigma'}$ determine the same orientation at points $x \in \operatorname{strat}(\sigma) \cap U_{\tau,b}^{\tau'} \cap U_{\tau,a_\tau}^{\sigma'}$. Finally $\phi_{\sigma,a}$ and $\phi_{\tau,b}^{\tau'}$ determine the same orientation for all $x \in \operatorname{strat}(\sigma) \cap U_{\sigma,a} \cap U_{\tau,b}^{\tau'}$. $\square$

8.3. Resolution of singularities of toroidal varieties in arbitrary characteristic.
We call a Γ-semicomplex Σ *regular* if all cones $|\sigma|$, where $\sigma \in \Sigma$ are regular.

Proposition 8.3.1. *For any oriented Γ-semicomplex Σ there exists a sequence of stable vectors $v_1, \ldots, v_m \in \operatorname{Stab}(\Sigma)$ such that the subdivision $\Delta := \langle v_k \rangle \cdot \ldots \cdot \langle v_1 \rangle \cdot \Sigma$ of Σ is a regular Γ-semicomplex.*

Proof. Let $\sigma_1, \ldots, \sigma_k$ denote all the faces of Σ. We shall construct by induction on i the canonical subdivision Δ_i of Σ such that the fan $\Delta_i^{\sigma_j}$ is regular for all $j \leq i$. Suppose Δ_i is already constructed. By [39] or [20] there exist centers $v_1^i, \ldots, v_{k_i}^i$ for the fan $\Delta_i^{\sigma_{i+1}}$ in $N_{\sigma_{i+1}}$ such that $\langle v_{k_i}^i \rangle \cdot \ldots \cdot \langle v_1^i \rangle \cdot \Delta_i^{\sigma_{i+1}}$ is regular. The subsequent centers v_j^i of star subdivisions coul be chosen to be minimal internal vectors in indecomposable faces of $\langle v_{j-1}^i \rangle \cdot \ldots \cdot \langle v_1^i \rangle \cdot \Delta_i^{\sigma_{i+1}}$. Therefore by Lemma 5.3.16 all centers in the desingularization process are Σ-stable. Hence by Proposition 5.4.1, $\Delta_{i+1} := \langle v_k^i \rangle \cdot \ldots \cdot \langle v_1^i \rangle \cdot \Delta_i$ is canonical. By construction $\Delta_{i+1}^{\sigma_{i+1}}$ is regular. Note that $\Delta_{i+1}^{\sigma_j} = \Delta_i^{\sigma_j}$ for $j < i + 1$ since during the desingularization regular cones remain unaffected. $\square$

Theorem 8.3.2. *For any Γ-stratified toroidal variety (X, S) (respectively a Γ-toroidal variety X) there exists a sequence of Γ-equivariant blow-ups at locally monomial valuations $\operatorname{bl}_{\nu_k} \circ \ldots \circ \operatorname{bl}_{\nu_1}(X)$ which is a resolution of singularities.*

Proof. In the case of a nonstratified toroidal variety consider X with stratification $\operatorname{Sing}^\Gamma(X)$. By Lemma 8.2.6 we can assume that the Γ-stratified toroidal variety is oriented. By Lemma 8.2.3 the associated semicomplex is oriented. By Proposition 8.3.1 and Proposition 5.4.1, $\operatorname{bl}_{\nu_k} \circ \ldots \circ \operatorname{bl}_{\nu_1}(X)$ is smooth. $\square$

9. Orientation of toroidal modifications

9.1. Lifting group actions.
Let U be an affine variety. Let $\widehat{X}_x$ be a completion of a variety X at its closed point x. Let $\Theta : Y \to X$ be a proper morphism and $\widehat{Y}_x := \widehat{X}_x \times_X Y$.

Set
$$U \widehat{\times} \widehat{X}_x := \operatorname{Spec}(\varprojlim K[U] \otimes \mathcal{O}_x^n / m_{X,x}),$$
$$U \widehat{\times} \widehat{Y}_x := (U \widehat{\times} \widehat{X}_x) \times_X Y.$$

Lemma 9.1.1. *If U, X are normal varieties then the scheme $U \widehat{\times} \widehat{X}_x$ is normal. If U, Y are normal varieties then the scheme $U \widehat{\times} \widehat{Y}_x$ is normal.*

Proof. (1) $\pi_X : U \widehat{\times} \widehat{X}_x \to U \times \widehat{X}_x$ is étale at (u, x) for $u \in U$ since it determines an isomorphism of the completions of the local rings at (u, x). $U \times \widehat{X}_x$ is normal therefore $U \widehat{\times} \widehat{X}_x$ is normal at $(u, x) \in U \times \{x\}$. But all closed points of $U \widehat{\times} \widehat{X}_x$ are in $U \times \{x\} \subset U \widehat{\times} \widehat{X}_x$. The morphism π is étale in a neighborhood of $U \widehat{\times} \widehat{X}_x$. Such a neighborhood is equal to $U \widehat{\times} \widehat{X}_x$ since its complement if nonempty would contain some closed points.

(2) $\pi_Y : U \widehat{\times} \widehat{Y}_x \to U \times \widehat{Y}_x$ is étale at (u, y) for $u \in U$ and $y \in \theta^{-1}(x)$ since π_Y is a pull-back of π_X. The rest of the reasoning is the same. $\square$

Definition 9.1.2. Let a proalgebraic group G act on $\widehat{X}_x$. Let $\Phi : G \widehat{\times} \widehat{X}_x \to \widehat{X}_x$ be the action morphism and $\Psi : G \widehat{\times} \widehat{X}_x \to G \widehat{\times} \widehat{X}_x$ be the action automorphism (see Lemma 4.5.1). Let $\theta : U \to G$ be a morphism from a normal algebraic variety U to G.

Then by the *action morphism with respect to U* or simply *action morphism* we mean the induced morphism $\Phi_{U, \widehat{X}_x} : U \widehat{\times} \widehat{X}_x \to \widehat{X}_x$.

By the *action automorphism with respect to U* or simply *action automorphism* we mean the induced morphism $\Psi_{U, \widehat{X}_x} : U \widehat{\times} \widehat{X}_x \to U \widehat{\times} \widehat{X}_x$.

Let $\alpha : Y \to \widehat{X}_x$ be a G^K-equivariant birational morphism. By the *lifting* of the action morphism (resp. automorphism) we mean the morphism $\Phi_{U,Y} : U \widehat{\times} Y \to Y$ (resp. the automomorphism $\Psi_{U,Y} : U \widehat{\times} Y \to U \widehat{\times} Y$) commuting with the induced morphism $\operatorname{id} \widehat{\times} \alpha : U \widehat{\times} Y \to U \widehat{\times} \widehat{X}_x$ and with α.

Lemma 9.1.3. *Let G be a proalgebraic group acting on $\widehat{X}_x$. Let $U \to G$ be any morphism from a normal algebraic variety U. Let I be a G-invariant ideal on $\widehat{X}_x$. Let $\phi : Y \to \widehat{X}_x$ be the normalization of the blow-up at I. Then there is a lifting of the action automorphism (and action morphism) $\Psi_{U,\widehat{X}_x} : U \widehat{\times} \widehat{X}_x \to U \widehat{\times} \widehat{X}_x$ to $\Psi_{U,Y} : U \widehat{\times} Y \to U \widehat{\times} Y$.*

Proof. Let $\Phi := \Phi_{U,\widehat{X}_x} : U \widehat{\times} \widehat{X}_x \to \widehat{X}_x$ denote the action morphism. Set $\Psi := \Psi_{U,\widehat{X}_x}$. By Lemma 4.5.2 we know that $J =: p^*(I) = \Phi^*(I) = \Psi^* p^*(I) = \Psi^*(J)$ is preserved by the action automorphism. Therefore J lifts to the blow-up of $U \widehat{\times} \widehat{X}_x$ at J and to its normalization. Since $U \widehat{\times} \widehat{X}_x \to \widehat{X}_x$ is étale, the normalization of the blow-up of $U \widehat{\times} \widehat{X}_x$ at J is isomorphic to $U \widehat{\times} Y$. $\square$

Proposition 9.1.4. *Let σ be a cone of maximal dimension in N_σ. Let G be a connected proalgebraic group acting on $X := \widehat{X}_\sigma$. Let $g : U \to G$ be a morphism from a normal algebraic variety U. Let G^K act on $Y := \widehat{X}_\Sigma$ such that $\psi : Y \to X$ is a G^K-equivariant proper birational morphism. Then there is a lifting of the action automorphism $\Psi_{U,X} : U \widehat{\times} X \to U \widehat{\times} X$ to $\Psi_{U,Y} : U \widehat{\times} Y \to U \widehat{\times} Y$.*

Proof. Let $\Psi_{U,Y} : U \times Y - \to U \times Y$ be a birational map which is a lifting of $\Phi_{U,Y}$. By Lemmas 7.6.8 and 9.1.3 we can find a factorization $Z \to Y \to X$ giving the diagram of proper morphisms

$$
\begin{array}{ccccc}
U \widehat{\times} Z & \xrightarrow{\Psi_{U,Z}} & & & U \widehat{\times} Z \\
\downarrow \searrow & & W & & \nearrow \downarrow \\
& \nearrow f_1 & \xrightarrow[\Psi_{U,Y}]{} & \searrow f_2 & \\
U \widehat{\times} Y & & & & U \widehat{\times} Y \\
\downarrow \phi_{U \widehat{\times} Y} & & & & \downarrow \phi_{U \widehat{\times} Y} \\
U \widehat{\times} X & \xrightarrow[\Psi_{U,X}]{} & & & U \widehat{\times} X
\end{array}
$$

where W is the irreducible component of the fiber product of $U \widehat{\times} Y \times_{U \widehat{\times} X} U \widehat{\times} Y$ which dominates $U \widehat{\times} Y$. (The fiber product is given by the morphisms $\phi_{U \widehat{\times} Y}$ and $\Psi_{U,X} \phi_{U \widehat{\times} Y}$.) Let $u \in U$ be a closed point. Take a pull-back of the above diagram via the closed embedding: $\{u\} \widehat{\times} X \to U \widehat{\times} X$

$$
\begin{array}{ccccc}
\{u\} \widehat{\times} Z & \xrightarrow{\Psi_{u,Z}} & & & \{u\} \widehat{\times} Z \\
\downarrow \searrow & & W_u & & \nearrow \downarrow \\
& \nearrow & \xrightarrow[\Psi_{u,Y}]{} & \searrow & \\
\{u\} \widehat{\times} Y & & & & \{u\} \widehat{\times} Y \\
\downarrow & & & & \downarrow \\
\{u\} \widehat{\times} X & \xrightarrow[\Psi_{u,X}]{} & & & \{u\} \widehat{\times} X
\end{array}
$$

The birational map $\Psi_{u,Y}$ is the isomorphism defined by the action of $g(u) \in G$ on Y. W_u consists of some components of the fiber product $\{u\} \widehat{\times} Y \times_{\{u\} \widehat{\times} X} \{u\} \widehat{\times} Y$. In particular it contains the irreducible component dominating $\{u\} \widehat{\times} Y$ which is isomorphic to Y. On the other hand $\{u\} \widehat{\times} Z \to W_u$ is a proper surjective morphism. Consequently, W_u is irreducible and isomorphic to Y. By Lemma 9.1.1, $U \widehat{\times} Y$ is normal. Thus $f_i : W \to U \widehat{\times} Y$ is a proper birational morphism onto a normal scheme which is bijective on the set of closed (K-rational) points. By the Zariski theorem f_i is an isomorphism and $\Psi_{U,Y} = f_1^{-1} f_2$ is an action automorphism. $\square$

9.2. Orientation of toroidal modifications.

Proposition 9.2.1. *Let (X, S) be an oriented Γ-stratified toroidal variety with the associated Γ-semicomplex Σ. Let $f : (Y, R) \to (X, S)$ be the toroidal morphism associated to a canonical subdivision Δ of Σ. Then*

1. *The stratified toroidal variety (Y, R) is <u>oriented</u> with charts from $\bigcup_{\sigma \in \Sigma} \phi_\sigma^{f*}(\mathcal{U}(\Delta^\sigma, \Delta_{\text{stab}}^\sigma))$ induced by the charts $\phi_\sigma : U \to X_\sigma$ on X and the associated oriented semicomplex $\Sigma_R = \Delta_{\text{stab}}$.*
2. *For any point x in a stratum $s \in S$ every Γ_s-equivariant automorphism α of $\widehat{X}_x$ preserving strata and orientation lifts to a Γ_s-equivariant automorphism α' of $Y \times_X \widehat{X}_x$ such that the atlases*

$\alpha'^* \hat{\phi}_\sigma^{f*}(\mathcal{U}(\Delta^\sigma, \Delta^\sigma_{\text{stab}}))$ and $\hat{\phi}_\sigma^{f*}(\mathcal{U}(\Delta^\sigma, \Delta^\sigma_{\text{stab}}))$ are compatible along $f^{-1}(x) \subset Y \times_X \widehat{X}_x$. Here $\hat{\phi}_\sigma^f :$ $Y \times_X \widehat{X}_x \to X_{\Delta^\sigma}$ denotes the morphism induced by ϕ_σ^f.

3. The stable support of the oriented Γ-semicomplex $\Sigma_R = \Delta_{\text{stab}}$ is contained in the stable support of Σ.

Proof.($\Leftarrow$) Let $\phi_{\sigma,1} : U_{\sigma,1} \to X_\sigma$ and $\phi_{\tau,2} : U_{\tau,2} \to X_\tau$ be two charts on X, where $\sigma \leq \tau$. For (1) we have to show that $\phi_{\sigma,1}^{f*}(\mathcal{U}(\Delta^\sigma, \Delta^\sigma_{\text{stab}}))$ and $\phi_{\tau,2}^{f*}(\mathcal{U}(\Delta^\tau, \Delta^\tau_{\text{stab}}))$ are compatible on $f^{-1}(U_{\sigma,1} \cap U_{\tau,2})$ along $f^{-1}(x)$ for any point $x \in \text{strat}(\sigma) \cap (U_{\sigma,1} \cap U_{\tau,2})$.

Let $\tau' \preceq \tau$ denote a face of τ such that $\omega(\tau') = \sigma$ and $\phi_{\tau,2}(x) \in O_{\tau'}$. Then $\tau' = \sigma \times r(\tau')$, where $r(\tau')$ denotes a regular cone and $\tau' \setminus \sigma$ is disjoint from $\text{Stab}(\Sigma)$. By Lemma 4.12.8, $\Delta^\tau|\tau' = \Delta^\tau|\sigma \oplus r(\tau') = \Delta^\sigma \times r(\tau')$. Also, $\Delta^\tau_{\text{stab}}|\tau' = \Delta^\sigma_{\text{stab}} \times \{0\}$.

Denote by $\phi_{\tau,2}^{\tau'} : U_{\tau,2}^{\tau'} \to X_{\tau'}$, the restriction of $\phi_{\tau,2}$ and by $\phi_{\tau,2}^{\tau'f}$ its lifting. Then $\phi_{\tau,2}^{\tau'f*}(\mathcal{U}(\Delta^\tau|\tau', \Delta^\tau_{\text{stab}}|\tau'))$ is the restriction of $\phi_{\tau,2}^{f*}(\mathcal{U}(\Delta^\tau, \Delta^\tau_{\text{stab}}))$ to the open set $f^{-1}(U_{\tau,2}^{\tau'})$ containing $f^{-1}(x)$. Set $U_{\sigma,2} := U_{\tau,2}^{\tau'}$, $\phi_{\sigma,2} := \pi_{\tau'}^\sigma \phi_{\tau,2}^{\tau'} : U_{\sigma,2} \to X_\sigma$, where $\pi_{\tau'}^\sigma : X_\tau' \to X_\sigma$ is defined by any projection which is identical on $\sigma \preceq \tau'$. Let $\phi_{\sigma,2}^f := \pi_{\tau'}^{\sigma f} \phi_{\tau,2}^{\tau'f} : f^{-1}(U_{\sigma,2}) \to X_\sigma$ be the lifting of $\phi_{\sigma,2}$. Then

$$\phi_{\tau,2}^{\tau'f*}(\mathcal{U}(\Delta^\tau|\tau', \Delta^\tau_{\text{stab}}|\tau')) = \phi_{\tau,2}^{\tau'f*}\mathcal{U}(\Delta^\sigma \times r(\tau'), \Delta^\sigma_{\text{stab}} \times \{0\})$$

is compatible along $f^{-1}(x)$ with

$$\phi_{\tau,2}^{\tau'f*}\pi_{\tau'}^{\sigma f*}(\mathcal{U}(\Delta^\sigma, \Delta^\sigma_{\text{stab}})) = \phi_{\sigma,2}^{f*}(\mathcal{U}(\Delta^\sigma, \Delta^\sigma_{\text{stab}})).$$

It suffices to show that $\phi_{\sigma,1}^{f*}(\mathcal{U}(\Delta^\sigma, \Delta^\sigma_{\text{stab}}))$ and $\phi_{\sigma,2}^{f*}(\mathcal{U}(\Delta^\sigma, \Delta^\sigma_{\text{stab}}))$ are compatible along $f^{-1}(x)$.

Let $U \subset U_{\sigma,1} \cap U_{\sigma,2}$, be an open subset for which there exist étale extensions $\widetilde{\phi}_i : U \to X_{\widetilde{\sigma}}$ of $\phi_{\sigma,i}$, $i = 1, 2$. Assume that $\widetilde{\phi}_i(x) = O_{\widetilde{\sigma}} = O_{\sigma \times \text{reg}(\sigma)}$, where $\text{reg}(\sigma)$ is a regular cone of dimension $k = \dim(\text{strat}(\sigma))$. Let $\widetilde{\phi}_{fi}$ be an étale extension of ϕ_{fi} which is a lifting of $\widetilde{\phi}_i$.

Consider the fiber squares

$$
\begin{array}{ccccc}
\widetilde{\phi}_i : & U & \to & X_{\widetilde{\sigma}} & = X_{\sigma \times \text{reg}(\sigma)} \\
& \uparrow & & \uparrow f_{\widetilde{\Delta}} & \\
\widetilde{\phi}_{fi} : & f^{-1}(U) & \to & X_{\widetilde{\Delta}^\sigma} & = X_{\Delta^\sigma \times \text{reg}(\sigma)}
\end{array}
\qquad
\begin{array}{ccccc}
\widehat{\phi}_i : & \widehat{X}_x & \to & \widetilde{X}_\sigma \\
& \uparrow & & \uparrow \widetilde{f}_\Delta \\
\widehat{\phi}_{fi} : & Y \times_X \widehat{X}_x & \to & \widetilde{X}_{\Delta^\sigma}
\end{array}
$$

The morphism $\widetilde{f}_\Delta$ is a pull-back of the morphism $f_{\widetilde{\Delta}}$. Therefore $\widetilde{f}_\Delta^{-1}(O_{\widetilde{\sigma}})$ is a K-subscheme of $\widetilde{X}_{\Delta^\sigma}$.

The automorphism $\widehat{\phi} := \widehat{\phi}_2 \widehat{\phi}_1^{-1}$ of $\widetilde{X}_\sigma \simeq \widehat{X}_x$ preserves strata and orientation. For the proof of conditions (1) and (2) we need to show that any such automorphism $\widehat{\phi}$ preserving strata and orientation lifts to the automorphism $\widehat{\phi}^f = \widehat{\phi}_2^f(\widehat{\phi}_1^f)^{-1}$ of $\widetilde{X}_{\Delta^\sigma} \simeq Y \times_X \widehat{X}_x$ such that $\widehat{\phi}^{f*}(\mathcal{U}(\Delta^\sigma, \Delta^\sigma_{\text{stab}}))$ and $\mathcal{U}(\Delta^\sigma, \Delta^\sigma_{\text{stab}})$ are compatible along $\widetilde{f}_\Delta^{-1}(O_{\widetilde{\sigma}_\sigma})$.

By Lemma 7.3.1 there is an action morphism $\Phi : W \widehat{\times} \widetilde{X}_\sigma \to \widetilde{X}_\sigma$ such that

1. $\Phi_e := \Phi_{|e\widehat{\times}\widetilde{X}_\sigma} = \text{id}_{X_\sigma}$.
2. There exists $g \in W$ such that $\Phi_g := \Phi_{|g\widehat{\times}\widetilde{X}_\sigma} = \widehat{\phi}$.
   By Proposition 9.1.4, Φ can be lifted to $\Phi^f : W \widehat{\times} \widetilde{X}_{\Delta^\sigma} \to \widetilde{X}_{\Delta^\sigma}$ such that
3. $\Phi_e^f = \text{id}_{\widetilde{X}_{\Delta^\sigma}}$.
4. $\Phi_g^f = \widehat{\phi}^f$.

Let $\Pi : W \widehat{\times} \widetilde{X}_\sigma \to \widetilde{X}_\sigma$ and $\Pi^f : W \widehat{\times} \widetilde{X}_{\Delta^\sigma} \to \widetilde{X}_{\Delta^\sigma}$ denote the natural projections.

Note that $\Pi^{-1}(O_{\widetilde{\sigma}}) = \Phi^{-1}(O_{\widetilde{\sigma}}) = W \widehat{\times} O_{\widetilde{\sigma}} \simeq W \times O_{\widetilde{\sigma}} \simeq W$. The morphisms Π^f and Φ^f are pull-backs of Π and Φ induced by $\widetilde{f}_\Delta$. Thus $(\Pi^f)^{-1}(f_{\widetilde{\Delta}}^{-1}(O_{\widetilde{\sigma}})) = (\Phi^f)^{-1}(f_{\widetilde{\Delta}}^{-1}(O_{\widetilde{\sigma}}) = W \widehat{\times} f_{\widetilde{\Delta}}^{-1}(O_{\widetilde{\sigma}}) \simeq W \times f_{\widetilde{\Delta}}^{-1}(O_{\widetilde{\sigma}})$ is a K-subscheme of $W \widehat{\times} \widetilde{X}_{\Delta^\sigma}$.

For the sake of our considerations we shall enrich the stratification R by adding to strata $r \in R$ their intersections with $f^{-1}(x)$. Set $\overline{R} = \{r \setminus f^{-1}(x) \mid r \in R\}\} \cup \{r \cap f^{-1}(x) \mid r \in R\}$.

Strata of R on $f^{-1}(U)$ correspond to the embedded semifan $\Delta^\sigma_{\text{stab}} \subset \Delta^\sigma \times \text{reg}(\sigma)$. Strata of $\overline{R}$ correspond to the embedded semifan $\Omega \subset \Delta^\sigma \times \text{reg}(\sigma)$, where $\Omega := \Delta^\sigma_{\text{stab}} \cup \{\omega \times \text{reg}(\sigma) \mid \omega \in \Delta^\sigma_{\text{stab}}, \text{int}(\omega) \subset \text{int}(\sigma)\}$.

For any stratum $\mathrm{strat}_Y(\omega) \in R$, $\omega \in \Delta^\sigma_{\mathrm{stab}}$, dominating the stratum $\mathrm{strat}_X(\sigma)$ let $\mathrm{strat}(\omega)$ denote the corresponding stratum on $\widetilde{X}_{\Delta^\sigma}$. Let $\widetilde{\omega} := \omega \times \mathrm{reg}(\sigma) \in \Omega$. Then $\mathrm{strat}(\omega) \cap f_{\widetilde{\Delta}}^{-1}(O_{\widetilde{\sigma}}) = \mathrm{strat}(\widetilde{\omega}) \subset f_{\widetilde{\Delta}}^{-1}(O_{\widetilde{\sigma}}) \subset \widetilde{X}_{\Delta^\sigma}$ is a G^σ-invariant stratum corresponding to $\widetilde{\omega} \in \Omega$. Since $\mathrm{strat}(\widetilde{\omega})$ is an irreducible locally closed subset of $f_{\widetilde{\Delta}}^{-1}(O_{\widetilde{\sigma}})$ it is an algebraic variety.

The natural isomorphism $(\Phi^f)^{-1}(\mathrm{strat}(\widetilde{\omega})) = (\Pi^f)^{-1}(\mathrm{strat}(\widetilde{\omega})) = W \widehat{\times} \mathrm{strat}(\widetilde{\omega}) \longrightarrow W \times \mathrm{strat}(\widetilde{\omega})$ is a pull-back of the isomorphism $W \widehat{\times} O_{\widetilde{\sigma}} \longrightarrow W \times O_{\widetilde{\sigma}}$ induced by $f_{\widetilde{\Delta}}^{-1}(O_{\widetilde{\sigma}}) \to O_{\widetilde{\sigma}}$.

Thus $W \widehat{\times} \mathrm{strat}(\widetilde{\omega}) = W \times \mathrm{strat}(\widetilde{\omega})$ is an algebraic variety.

By definition we have $\Psi^f_e = \Pi^f_e = \mathrm{id}_{|\mathrm{strat}(\widetilde{\omega})}$. Hence by Lemma 4.10.5 applied to the natural projection of $W \widehat{\times} \mathrm{strat}(\widetilde{\omega})$ on W the collections of charts $\Psi^{f*}(\mathcal{U}^{\mathrm{can}}(\widetilde{X}_{\Delta^\sigma}, S_\Omega))$ and $\Pi^{f*}(\mathcal{U}^{\mathrm{can}}(\widetilde{X}_{\Delta^\sigma}, S_\Omega))$ are compatible along $\{e\} \widehat{\times} \mathrm{strat}(\widetilde{\omega})$.

Hence by Lemma 8.2.3, $\Psi^{f*}(\mathcal{U}^{\mathrm{can}}(\widetilde{X}_{\Delta^\sigma}, S_\Omega))$ and $\Pi^{f*}(\mathcal{U}^{\mathrm{can}}(\widetilde{X}_{\Delta^\sigma}, S_\Omega))$ are compatible along $\{g\} \widehat{\times} \mathrm{strat}(\widetilde{\omega})$. Again by Lemma 4.10.5, $\Psi^{f*}_g(\mathcal{U}^{\mathrm{can}}(\widetilde{X}_{\Delta^\sigma}, S_\Omega)) = \widehat{\psi}^{f*}(\mathcal{U}^{\mathrm{can}}(\widetilde{X}_{\Delta^\sigma}, S_\Omega))$ and $\Pi^{f*}_g(\mathcal{U}^{\mathrm{can}}(\widetilde{X}_{\Delta^\sigma}, S_\Omega)) = \mathcal{U}^{\mathrm{can}}(\widetilde{X}_{\Delta^\sigma}, S_\Omega)$ are compatible along $\mathrm{strat}(\widetilde{\omega})$. The charts $\phi_{\omega'}$, $\omega' \geq \omega$, of $\mathcal{U}^{\mathrm{can}}(\widetilde{X}_{\Delta^\sigma}, S_{\Delta^\sigma_{\mathrm{stab}}})$ are obtained by composing the charts $\phi_{\widetilde{\omega'}}$ of $\mathcal{U}^{\mathrm{can}}(\widetilde{X}_{\Delta^\sigma}, S_\Omega)$ with the natural projection $X_{\widetilde{\omega'}} \to X_{\omega'}$. Thus the atlases $\mathcal{U}^{\mathrm{can}}(\widetilde{X}_{\Delta^\sigma}, S_{\Delta^\sigma_{\mathrm{stab}}})$ and $\widehat{\psi}^{f*}(\mathcal{U}^{\mathrm{can}}(\widetilde{X}_{\Delta^\sigma}, S_{\Delta^\sigma_{\mathrm{stab}}})$ are compatible along $\mathrm{strat}(\omega) \cap f_{\widetilde{\Delta}}^{-1}(O_{\widetilde{\sigma}}) = \mathrm{strat}(\widetilde{\omega})$.

(3) By Lemma 8.2.5, $\Sigma_R = \Delta_{\mathrm{stab}}$ is an oriented Γ-semicomplex and the notion of stability makes sense for Σ_R. Let $v \in \mathrm{int}(\omega)$, where $\omega \in \Delta^\sigma_{\mathrm{stab}}$, be a Σ_R-stable vector. Assume that $\mathrm{int}(\omega) \subset \mathrm{int}(\sigma)$. We need to show that for any $\tau \geq \sigma$, any automorphism α of $\mathrm{Aut}(\widetilde{X}_\tau)^0$ preserves $\mathrm{val}(v, \widetilde{X}_\tau)$. Find the cone $\omega' \in \Delta^\tau$ such that $\omega' \geq \omega$ and $\mathrm{int}(\omega') \subset \mathrm{int}(\tau)$. By the convexity of $\mathrm{stab}(\tau)$, $\mathrm{int}(\omega')$ intersects $\mathrm{stab}(\tau)$ and $\omega' \in \Delta^\tau_{\mathrm{stab}}$.

Denote by $\pi : \widetilde{X}_{\Delta^\tau} \to X_{\Delta^\tau}$ the standard projection. By (2) α lifts to an automorphism α^f of $\widetilde{X}_{\Delta^\tau}$ such that $\alpha^{f*}\pi^*(\mathcal{U}(\Delta^\tau, \Delta^\tau_{\mathrm{stab}})$ and $\pi^*(\mathcal{U}(\Delta^\tau, \Delta^\tau_{\mathrm{stab}})$ are compatible along $f_\Delta^{-1}(O_{\widetilde{\tau}})$.

Let $p := O_{\omega' \times \mathrm{reg}(\tau)} = O_{\omega'} \cap f_\Delta^{-1}(O_{\widetilde{\tau}}) \subset \widetilde{X}_{\Delta^\tau}$. Set $Y := \widetilde{X}_{\Delta^\tau}$. By Lemma 4.7.3, $\widehat{Y}_p = \widehat{X}_{\omega' \times reg(\tau)} = \widetilde{X}_{\omega'}$. By assumption $\mathrm{val}(v, \widehat{Y}_p)$ is invariant with respect to any $\Gamma_{\omega'} = \Gamma_\tau$-equivariant automorphisms of $\widehat{Y}_p = \widetilde{X}_{\omega'}$ preserving orientation and strata. The automorphism α^f preserves p and induces an automorphism $\widehat{\alpha}^f_p$ of $\widehat{Y}_p$. Denote by $\widehat{\pi} : \widehat{Y}_p \to X_{\omega'}$ the morphism induced by π. Since the atlases $\alpha^{f*}_p \pi^*(\mathcal{U}(\Delta^\tau, \Delta^\tau_{\mathrm{stab}})$ and $\pi^*(\mathcal{U}(\Delta^\tau, \Delta^\tau_{\mathrm{stab}}))$ are compatible the morphisms $\widehat{\pi}\widehat{\alpha}^f_p : \widehat{Y}_p \to X_{\omega'}$ and $\widehat{\pi}$ determine the same orientation at p. Thus $\widehat{\alpha}^f_p$ preserves orientation at p and $\widehat{\alpha}^f_{p*}(\mathrm{val}(v, Y_p)) = \mathrm{val}(v, Y_p)$. Then also $\alpha^f_*(\mathrm{val}(v, \mathrm{Spec}(\mathcal{O}_{Y,p})) = \mathrm{val}(v, \mathrm{Spec}(\mathcal{O}_{Y,p})$. Consequently, $\alpha^f_*(\mathrm{val}(v, \widetilde{X}_{\Delta^\tau}) = \mathrm{val}(v, \widetilde{X}_{\Delta^\tau})$ and $\alpha_*(\mathrm{val}(v, \widetilde{X}_\tau)) = \mathrm{val}(v, \widetilde{X}_\tau)$. $\square$

10. The π-desingularization lemma of Morelli

10.1. Local projections of Γ-semicomplexes.

Definition 10.1.1. Let Σ be a simplicial Γ-semicomplex and Δ be its canonical subdivision. Let $\pi_\sigma : \sigma \to \sigma^\Gamma$ be the projection defined by the quotient map $X_\sigma \to X_\sigma/\Gamma_\sigma = X_{\sigma^\Gamma}$, for any $\sigma \in \Sigma$. For any $\delta \in \Delta^\sigma$ set $\Gamma_\delta = (\Gamma_\sigma)_\delta = \{g \in \Gamma_\sigma \mid gx = x, x \in O_\delta\}$. By $\pi_\delta : \delta \to \delta^\Gamma$ denote the projection defined by the quotient map $X_\delta \to X_\delta/\Gamma_\delta = X_{\delta^\Gamma}$.

We say that Σ is *strictly π-convex* if for any $\sigma \in \Sigma$, $\pi_\sigma(\sigma) = \sigma^\Gamma$ is strictly convex (contains no line). We call a semicomplex Σ *simplicial* if all cones $|\sigma|$, $\sigma \in \Sigma$, are simplicial.

Lemma 10.1.2.   1. *Let Σ be a strictly convex Γ-semicomplex. Then for any $\tau \leq \sigma$, there exists an inclusion $\iota^{\sigma\Gamma}_\tau : \tau^\Gamma \hookrightarrow \sigma^\Gamma$ and the commutative diagram of inclusions:*

$$
\begin{array}{ccccc}
\iota^\sigma_\tau : & \tau & \hookrightarrow & \sigma \\
& \pi_\tau \downarrow & & \pi_\sigma \downarrow \\
\iota^{\sigma\Gamma}_\tau : & \tau^\Gamma & \hookrightarrow & \sigma^\Gamma
\end{array}
$$

2. *Let Δ be a canonical subdivision of Σ. Then for any $\delta \in \Delta^\sigma_{\mathrm{stab}}$ we have the commutative diagram of inclusions*

$$
\begin{array}{ccc}
\delta & \subset & \sigma \\
\pi_\delta \downarrow & & \pi_\sigma \downarrow \\
\delta^\Gamma & \hookrightarrow & \sigma^\Gamma
\end{array}
$$

Proof. (1) Consider the commutative diagram

$$
\begin{array}{ccc}
X_{(\tau,N_\sigma)} & \simeq & X_\tau \times T \\
\downarrow & & \downarrow \\
X_{(\tau,N_\sigma)}//\Gamma_\tau & \overset{i}{\simeq} & X_\tau//\Gamma_\tau \times T \\
\downarrow & & \downarrow \\
X_{\pi_\sigma(\tau,N_\sigma)} = X_{(\tau,N_\sigma)}/\Gamma_\sigma & \overset{j}{\simeq} & X_{(\tau,N_\sigma)}//\Gamma_\sigma \times T/\Gamma_\sigma & \simeq & X_{(\tau^\Gamma,N_\sigma^\Gamma)}
\end{array}
$$

where T is the relevant torus. Note that Γ_τ acts trivially on $\mathrm{strat}(\tau) \subset X_\sigma$, hence in particular on the torus T in $X_\tau \times T$. This gives the isomorphism i. Since $\Gamma_\sigma/\Gamma_\tau$ acts freely on $X_{(\tau,N_\sigma)}//\Gamma_\tau = X_\tau//\Gamma_\sigma \times T$ we get the isomorphism j. Consequently, $(\tau^\Gamma, N_\sigma^\Gamma) \simeq \pi_\sigma(\tau, N_\sigma) \subset \sigma^\Gamma$.

(2) Repeat the reasoning from (1) with appropriate inclusions. $\qquad\square$

10.2. **Dependent and independent cones.** In further considerations we shall assume $\Gamma = K^*$.

Lemma 10.2.1. *Let Σ be a simplicial strictly convex K^*-semicomplex. Let $\mathrm{Dep}(\Sigma) := \{\sigma \in \Sigma \mid \Gamma_\sigma = K^*\}$ and $\mathrm{Ind}(\Sigma) := \{\sigma \in \Sigma \mid \Gamma_\sigma \neq K^*\}$. Then*

1. *For $\sigma \in \mathrm{Ind}(\Sigma)$, π_σ is an immersion into the lattice N_σ^Γ of the same dimension.*
2. *For $\sigma \in \mathrm{Dep}(\Sigma)$, π_σ is a submersion onto the lattice N_σ^Γ such that $\dim(N_\sigma) = \dim(N_\sigma^\Gamma) + 1$. Moreover there is a vector $v_\sigma \in N_\sigma$ determined by a K^*-action which spans the kernel of π_σ.*

Proof. If $\sigma \in \mathrm{Ind}(\Sigma)$ then Γ_σ is finite and there is an immersion $M_\sigma^\Gamma \to M_\sigma$, of lattices of the same dimension, where M^Γ is a lattice of Γ-invariant characters. The dual morphism $N_\sigma \to N_\sigma^\Gamma$ is also an immersion of lattices of the same dimension. If $\sigma \in \mathrm{Dep}(\Sigma)$ then $\Gamma_\sigma = K^*$ is a subtorus corresponding to the sublattice $N := (\mathbf{Q} \cdot v_\sigma) \cap N_\sigma$ of N_σ. The projection π_σ is defined by the quotient map $N_\sigma \to N_\sigma/N \simeq N_\sigma^\Gamma$. $\quad\square$

Lemma 10.2.2. *Let Σ be a simplicial strictly convex K^*-semicomplex. Then for any faces $\tau \leq \sigma$ in $\mathrm{Dep}(\Sigma)$, $\imath_\tau^\sigma(v_\tau) = v_\sigma$.*

Proof. The induced toric morphism $X_\tau \to X_\sigma$ is K^*-equivariant. $\qquad\square$

Definition 10.2.3. (see Morelli [49]) Let Δ be a canonical subdivision of a K^*-semicomplex Σ. Let $\sigma \in \Sigma$.

1. A cone $\delta \in \Delta^\sigma$ is called *independent* if the restriction of π_σ to δ is a lattice immersion. Otherwise $\delta \in \Delta^\sigma$ is called *dependent*.
2. A minimal dependent face of Δ^σ is called a *circuit*.
3. We call an independent face τ *up-definite* (respectively *down-definite*) with respect to a dependent face δ if there exists a nonzero functional F on $\delta \subset N_\sigma^\mathbf{Q}$ such that $F(v_\sigma) > 0$ (respectively $F(v_\sigma) < 0$), and $\tau = \{v \in \sigma \mid F(v) = 0\}$.

Lemma 10.2.4. *Each dependent cone $\delta = \langle v_1, \ldots, v_k \rangle \in \Delta^\sigma$ defines a unique linear dependence relation*

$$r_1 \pi_\sigma(v_1) + \ldots + r_k \pi_\sigma(v_k) = 0 \qquad\qquad *.$$

This relation is determined up to proportionality.

Proof. $\pi_\sigma(\delta)$ is $k-1$- dimensional cone spanned by k rays. $\qquad\square$

Definition 10.2.5. 1. The relation (*) is *positively normalized* if the rays $\langle v_i \rangle$ for which $r_i > 0$ form an up-definite face
2. The rays for which $r_i > 0$ (in a positively normalized relation) are called *positive*, the rays for which $r_i < 0$ are called *negative*, the rays for which $r_i = 0$ are called *null* rays. The face τ of a cone δ is called *codefinite* if it contains only positive or only negative rays.
3. For any cone δ we denote by δ_- (respectively δ_+) the fan consisting of all faces of δ which are down-definite (respectively up-definite). By δ^- (respectively δ^+) we denote the face of δ spanned by all negative rays and null rays (respectively by all positive rays and null rays).

Lemma 10.2.6. *The relation (*) is positively normalized iff there exists $\alpha > 0$, such that*

$$r_1 v_1 + \ldots + r_k v_k = \alpha v_\delta$$

Proof. Let F be a nonnegative functional on δ which is 0 exactly on $\langle v_i \mid r_i > 0\rangle$. Then $\langle v_i \mid r_i > 0\rangle$ is up-definite iff $F(v_\delta) > 0$. The latter is equivalent to $\alpha > 0$. $\square$

Lemma 10.2.7. *Let* $\tau = \langle v_1, \ldots, \check{v}_i, \ldots, v_k\rangle$ *be a face of* δ *of codimension* 1. *Then* τ *is up-definite iff* $r_i > 0$.

Proof. Let F be a nonnegative functional on δ which is 0 exactly on τ. Then by Lemma 10.2.6, $r_i > 0$ iff $F(v_\delta) > 0$. The latter means that τ is up-definite. $\square$

Lemma 10.2.8. *If* $\tau \preceq \delta$ *is up-definite (respectively down-definite) then any face* $\tau' \preceq \tau$ *is up-definite (respectively down-definite) with respect to* δ. $\square$

Proof. Let F be a nonnegative functional on δ which is 0 exactly on τ and such that $F(v_\delta) > 0$. Let F' be a nonnegative functional on τ which is 0 exactly on τ'. Then $nF + F'$, where $n >> 0$, is a nonnegative functional on δ which is 0 exactly on τ' and $(nF + F')(v_\delta) > 0$. $\square$

Lemma 10.2.9. *The set* δ_+ *(respectively* δ_-*) is a subfan of* $\overline{\delta}$ *with maximal faces of the form* $\langle v_1, \ldots, \check{v}_i, \ldots, v_k\rangle$, *where* $r_i > 0$ *(respectively* $r_i < 0$*). In particular, each boundary face is up-definite or down-definite. The projection* π_δ *maps* δ_+ *(respectively* δ_-*) onto the subdivision* $\pi_\delta(\delta_+)$ *(respectively* $\pi_\delta(\delta_-)$ *of* $\pi_\delta(\delta)$. *Moreover the restriction of* π *to any boundary face is a linear isomorphism.*

Proof. Let v be a vector in $|\delta^\Gamma|$. Then either the line $\pi^{-1}(v) = \{v\} + \lin(v_\delta)$ intersects the relative interiors of exactly two boundary faces at one point each and $\pi^{-1}(v) \cap \delta$ is an interval or $\pi^{-1}(v)$ intersects the relative interior of one boundary face at a point and $\pi^{-1}(v) \cap \delta$ is the point. In the first case one of the faces is up-definite and one is down-definite. In the second case let τ be the boundary face containing $p = \pi^{-1}(v) \cap \delta$. Let F be a nonnegative functional on δ^Γ which is zero exactly on τ. Let F' be any functional on $N_\delta^{\mathbf{Q}}$ which equals 0 on τ and such that $F'(v_\delta) \geq 0$. Then $(nF \circ \pi) \pm F'$, where $n >> 0$, is nonnegative on δ and equals 0 exactly on τ. Moreover $(nF \circ \pi + F')(v_\delta) > 0$ and $(nF \circ \pi - F')(v_\delta) < 0$. Hence τ is up-definite and down-definite. $\square$

Lemma 10.2.10. *Let* $\delta \in \Delta^\sigma$ *be a dependent cone. Set* $\delta^+ := \langle v_i \mid r_i > 0\rangle$. *The ideal* $I_{\overline{O}_{\delta^+}}$ *of the closure of* O_{δ^+} *is generated by the set of all characters with positive* K^**-weights.*

Proof. Let x^m, $m \in \delta^\vee \cap M_\delta$ be the character with the positive weight. Then $(m, v_\delta) > O$. By Lemma 10.2.6, there is $r_i > 0$ in a positively normalized relation such that $(m, v_i) > 0$. Then x^m is zero on the divisor corresponding to v_i and on O_{δ^+}. On the other hand for any $r_i > 0$ we can find a functional m_i such that $(m_i, v_i) > 0$, $(m_i, v_\delta) > O$, $(m_i, v_j) = 0$, where $j \neq i$. Then $\overline{O}_{\delta^+}$ is the set of the common zeros of x^{m_i}, $r_i > 0$. $\square$

We associate with a cone $\delta \in \Delta^\sigma$ and an integral vector $v \in \delta^\Gamma$ a vector $\mathrm{Mid}(v, \delta) \in \delta$ ([49]. If δ is independent face, then $\mathrm{Mid}(v, \delta)$ is defined to be the primitive vector of the ray $\mathbf{Q}_{\geq 0} \cdot \pi_\sigma^{-1}(v) \subset \delta$. If δ is dependent let $\pi_{\delta_-} = \pi_{\sigma|\delta_-}$ and $\pi_{\delta_+} = \pi_{\sigma|\delta_+}$ be the restrictions of π_σ to δ_- and δ_+. Then $\mathrm{Mid}(v, \delta)$ is the primitive vector of the ray spanned by $\pi_{\delta_-}^{-1}(v) + \pi_{\delta_+}^{-1}(v) \in \delta$.

Remark. By Lemma 10.1.2 the local projections $\pi_\sigma : N_\sigma \to N_\sigma^\Gamma$ commute with subdivisions and face restrictions therefore the above notions are coherent.

10.3. **Stable vectors on simplicial K^*-semicomplexes.** In the section Σ denotes an oriented strictly convex K^*-semicomplex.

Lemma 10.3.1. *Let* σ *be a semicone in* Σ. *Then* G_σ *acts on*
$$\widetilde{X}_{\sigma^\Gamma} := \widetilde{X}_\sigma / \Gamma_\sigma$$
and the morphism $\widetilde{X}_\sigma \to \widetilde{X}_{\sigma^\Gamma}$ *is* G_σ*-equivariant. Moreover* $\widetilde{X}_{\sigma^\Gamma} = \widehat{X}_{\sigma^\Gamma \times \mathrm{reg}(\sigma)}$.

Proof. Follows from the fact that the action of G^σ commutes with Γ_σ. $\square$

Lemma 10.3.2. *Let* σ *be a dependent cone in* Σ. *Then*

1. $\widetilde{X}_{\sigma_-} := X_{\sigma_-} \times_{X_\sigma} \widetilde{X}_\sigma \subset \widetilde{X}_\sigma$ *is* G_σ*-invariant.*
2. $\widetilde{X}_{\pi_{\sigma_-}(\sigma_-)} := X_{\pi_{\sigma_-}(\sigma_-)} \times_{X_{\pi(\sigma)}} \widetilde{X}_{\pi(\sigma)} \to \widetilde{X}_{\pi(\sigma)}$ *is proper* G_σ*-equivariant.*

Proof (1) By Lemma 10.2.9, $\sigma_- = \overline{\sigma} \setminus \mathrm{Star}(\sigma^+, \overline{\sigma})$. By Lemma 10.2.6, the ideal of $\overline{O}_{\sigma^+} = \widetilde{X}_\sigma \setminus \widetilde{X}_{\sigma_-}$ is generated by all semi-invariant functions with positive weights (see also [73], Example 2).

(2) Follows from Proposition 7.6.1 since $\mathrm{Vert}(\pi_{\sigma_-}(\sigma_-)) = \mathrm{Vert}(\pi(\sigma))$ $\qquad\square$

Lemma 10.3.3. *Let σ be a cone of the maximal dimension. Let F denote the functional on $\tau := \sigma^\vee$ defined by some integral vector $v \in N_\sigma$ which is not in σ. For any integral k set $\tau_k := \{v \in \tau \mid F(v) = k\}$. Then there are a finite number of vectors $w_{k1}, \ldots, w_{kl_k} \in \tau_k$ such that*

$$\tau_k = \bigcup_{i=1,\ldots,l_k} (w_{ki} + \tau_0).$$

Proof. We can replace τ by a regular cone τ_0 by considering the epimorphism $p : \tau_0 \to \tau$ mapping generators to generators. Then F defines a functional on τ_0. Let $\tau_{0k} := \{v \in \tau_0 \mid F(v) = k\}$. Let $x_1, \ldots, x_m$ define the standard coordinates on $\tau \simeq \mathbf{Z}^m_{\geq 0}$. Without loss of generality we can write $F = n_1 x_1 + \ldots + n_l x_l - n_{l+1} x_{l+1} - \ldots - n_r x_r$, where $l < r \leq m$ and all $n_i > 0$. Then $\tau_{0k} = (\tau_{0k} \cap (\mathbf{Z}^r_{\geq 0} \times \{0\})) + (\{0\} \times \mathbf{Z}^{m-r})$, where $\{0\} \times \mathbf{Z}^{m-r} \subset \tau_{00}$.

By a k-*minimal* vector we shall mean a vector $v \in \tau_{0k} \cap (\mathbf{Z}^r_{\geq 0} \times \{0\})$ such that there is no $w \in \tau_{00} \cap (\mathbf{Z}^r_{\geq 0} \times \{0\})$ for which $v - w \in \tau_{0k}$. It suffices to show that the number of k-minimal vectors is finite. Suppose $x_{i_0}(v) \geq (n_1 + \ldots + n_l)(n_{l+1} + \ldots + n_r) + |k|$ for some $i_0 \leq l$ and let $x_{j_0} := \max\{x_i(v) \mid l+1 \leq i \leq r\}$. Then
$x_{j_0} \geq \frac{n_1 x_1 + \ldots + n_l x_l}{n_1 + \ldots + n_l} = \frac{n_{l+1} x_{l+1} + \ldots + n_r x_r + k}{n_{l+1} + \ldots + n_r} \geq (n_1 + \ldots + n_l)(n_{l+1} + \ldots + n_r)/(n_{l+1} + \ldots + n_r) = n_1 + \ldots + n_l$

Then $w := (0, \ldots, n_{j_0}, \ldots, 0, \ldots, n_{i_0} \ldots 0) \in \tau_{00}$, where the i_0-th coordinate is n_{j_0} and the j_0-th coordinate is n_{i_0}. Thus $v - w \in \tau_{0k}$ and v is not k-minimal. Consequently, all minimal k-vectors satisfy $\max\{x_i(v) \mid 1 \leq i \leq r\} < (n_1 + \ldots + n_l)(n_{l+1} + \ldots + n_r) + |k|$. $\qquad\square$

As a corollary we get

Lemma 10.3.4. *Let $K[\widetilde{X}_\sigma] = \bigoplus K[\widetilde{X}_\sigma]^k$ be the decomposition according to weights with respect to the $\Gamma_\sigma = K^*$-action. Then $K[\widetilde{X}_\sigma]^k$ is a finitely generated $K[\widetilde{X}_\sigma]^0$-module.* $\qquad\square$

Lemma 10.3.5. *Let σ be a dependent cone in Σ. Denote by $j_{\pi_{\sigma_-}} : X_{\sigma_-} \to X_{\sigma_-}/\Gamma_\sigma$ and $\widetilde{j}_{\pi_{\sigma_-}} : \widetilde{X}_{\sigma_-} \to \widetilde{X}_{\sigma_-}/\Gamma_\sigma$ the quotient morphisms. Then*

1. $\widetilde{X}_{\sigma_-}/\Gamma_\sigma \simeq \widetilde{X}_{\pi_{\sigma_-}(\sigma_-)}$.
2. $\widetilde{j}_{\pi_{\sigma_-}} : \widetilde{X}_{\sigma_-} \to \widetilde{X}_{\pi_{\sigma_-}(\sigma_-)}$ *is G_σ-equivariant.*
3. *For any $v \in \pi(\sigma_-)$, $j_\pi^*(\mathrm{val}(v, X_{\pi_{\sigma_-}(\sigma_-)})) = \mathrm{val}(\pi_{\sigma_-}^{-1}(v), X_{\sigma_-})$.*
4. *If $\mathrm{val}(v, \widetilde{X}_{\pi_{\sigma_-}(\sigma_-)})$ is G^σ invariant then $\widetilde{j}_{\pi_{\sigma_-}}^*(\mathrm{val}(v, \widetilde{X}_{\pi_{\sigma_-}(\sigma_-)})) = \mathrm{val}(\pi_{\sigma_-}^{-1}(v), \widetilde{X}_{\sigma_-})$ is G^σ-invariant.*

Proof. (1) For $\tau \in \sigma_-$ set $\widetilde{X}_\tau := X_{(\tau, N_\sigma)} \times_{X_\sigma} \widetilde{X}_\sigma$,
$\widetilde{X}_{\pi_{\sigma_-}(\tau, N_\sigma)} := X_{\pi_{\sigma_-}(\tau)} \times_{X_{\pi(\sigma)}} \widetilde{X}_{\pi(\sigma)}$.

Then $K[\widetilde{X}_\tau] = K[X_{(\tau, N_\sigma)}] \otimes_{K[\widetilde{X}_\sigma]} K[\widetilde{X}_\sigma]$.

The elements of this ring are finite sums $\sum \chi_i f_i$, where $f_i \in K[\widetilde{X}_\sigma]$ and $\chi_i \in \tau^\vee$ is a character. Set
$R_1 := K[\widetilde{X}_\tau/\Gamma_\sigma] = K[\widetilde{X}_\tau^{\Gamma_\sigma}] = (K[X_\tau] \otimes_{K[X_\sigma]} K[\widetilde{X}_\sigma])^{\Gamma_\sigma}$,
$R_2 := K[\widetilde{X}_{\pi_{\sigma_-}(\tau)}] = K[X_{\pi_{\sigma_-}(\tau)}] \otimes_{K[X_{\pi(\sigma)}]} K[\widetilde{X}_{\pi(\sigma)}] \subset R_1$.

R_2 consists of elements $\sum \chi_i f_i$, where $\chi_i \in \tau^\vee$ and $f_i \in K[\widetilde{X}_\sigma]$ have weight 0. R_1 consists of elements $\sum \chi_i f_i$, where $\chi_i \in \tau^\vee$ and $f_i \in K[\widetilde{X}_\sigma]$ have opposite weights. By Lemma 10.3.4 there is a finite number of characters $\chi_{ij} \in \sigma^\vee$ such that $f_i = \sum \chi_{ij} f_{ij}$, where f_{ij} have weight zero. Finally, $\sum \chi_i f_i = \sum_i \chi_i(\sum_j \chi_{ij} f_{ij}) = \sum_i \sum_j (\chi_i \chi_{ij}) f_{ij} \in R_2$.

(2) Follows from (1).

(3) By Lemma 10.2.9 the projection $\pi := \pi_{\sigma_-}$ maps cones of σ_- isomorphically onto cones of $\pi(\sigma_-)$.

By [39], Ch. I Th. 9, the sheaf of ideals $\mathcal{I} := \mathcal{I}_{\mathrm{val}(\pi(v)),d}$ on $X_{\pi_{\sigma_-}(\sigma_-)}$ corresponds to the strictly convex piecewise linear function $\mathrm{ord}(\mathcal{I})$. By definition and Lemma 5.2.1, $\mathrm{ord}(\mathcal{I})$ equals 0 on all cones not containing v. Let $\tau \in \sigma_-$ be a cone containing v. By the proof of Lemma 5.2.8, if d is sufficiently divisible then for any face δ of τ such that δ does not contain v, there is an integral vector $m_{\pi(\delta), \pi(\tau)} \in \pi(\tau)^\vee$ for which $(m_{\pi(\delta), \pi(\tau)}, \pi(v)) = d$, $m_{\pi(\delta), \pi(\tau)}$ is 0 on $\pi(\delta)$ and $\mathrm{ord}(\mathcal{I})$ equals $m_{\pi(\delta), \pi(\tau)}$ on $\pi(\delta + \langle v \rangle)$.

In particular $\mathcal{I}$ is generated on each cone $X_{\pi(\tau)}$ by $m_{\pi(\delta),\pi(\tau)}$, where of δ is a face of τ that does not contain v. Let $m_\delta = m_{\pi(\delta)} \circ \pi \in \tau^\vee$ be the induced functional on M_τ. Thus the ideal $\pi^*(\mathcal{I})$ is generated on each cone X_τ, where $\tau \in \sigma_-$ contains v, by $m_{\delta,\tau}$, where δ doesn't contain v. Then $(m_{\delta,\tau}, v) = d$, m_δ is 0 on δ and $\mathrm{ord}(\pi^*(\mathcal{I}))$ equals $m_{\delta,\tau}$ on $\delta + \langle v \rangle$. This gives by the proof of Lemma 5.2.8, $\pi^*(\mathcal{I}_{\mathrm{val}(\pi(v)),d}) = \mathcal{I}_{\mathrm{val}(v),d}$.

(4) Follows from (2) and (3). $\square$

Lemma 10.3.6. *Let $\tau \leq \sigma$ be faces of Σ. Then either the inclusion $\imath_\tau^{\sigma\Gamma}$ maps τ^Γ to a face of σ^Γ, or τ is independent, σ is dependent, and τ^Γ is mapped isomorphically onto the face of $\pi(\sigma_-)$ (or $\pi(\sigma_+)$).*

Proof. If both faces τ and σ are independent then it follows that π_σ and π_τ are linear isomorphisms. Then $\imath_\tau^{\sigma\Gamma}(\tau^\Gamma)$ is a face of σ^Γ. If τ and σ are both dependent then the kernel of the projection π_σ is contained in $\mathrm{lin}(\tau)$. Then $\pi(\tau)$ is a face of $\pi(\sigma)$. If τ is an independent face of a dependent cone σ then τ is a face of σ_- or σ_+. Both fans consist of independent faces of σ and project onto the subdivisions $\pi(\sigma_-)$ and $\pi(\sigma_+)$ of $\pi(\sigma)$. $\square$

Definition 10.3.7. A vector $v \in \mathrm{int}(\sigma^\Gamma)$, where $\sigma \in \Sigma$, is Σ-*stable* if for any $\tau \geq \sigma$ the corresponding valuation $\mathrm{val}(\imath_\sigma^{\tau\Gamma}(v))$ on $\widetilde{X}_{\tau^\Gamma}$ is G^τ-invariant. A vector $v \in \sigma^\Gamma$ is Σ-*stable* if there is a Σ-stable vector $v_0 \in \mathrm{int}(\sigma_0^\Gamma)$, where $\sigma_0 \leq \sigma$, for which $v = \imath_{\sigma_0}^{\sigma\Gamma}(v_0)$.

Proposition 10.3.8. *Let σ be a semicone in Σ. A vector $v \in \mathrm{int}(\sigma)$ is Σ-stable if $\pi_\sigma(v) \in \sigma^\Gamma$ is Σ-stable.*

Proof. A stable vector $v \in \mathrm{int}(\sigma)$ determines a G^τ-invariant valuation on any $\widetilde{X}_\tau$ for $\tau \geq \sigma$. Consequently, it determines a G^τ-invariant valuation on any $\widetilde{X}_\tau/\Gamma_\tau$ and finally $\pi_\sigma(v)$ is Σ-stable.

Now let $v \in \mathrm{int}(\sigma)$ be an integral vector such that $\pi_\sigma(v) \in \mathrm{int}(\sigma^\Gamma)$ is stable. Then $\mathrm{val}(\pi_\sigma(v))$ is G^τ-invariant on $\widetilde{X}_\tau/\Gamma_\tau$. We have to prove that v is stable, or equivalently, that for any $\tau \geq \sigma$, $\mathrm{val}(v)$ is G^τ-invariant on $\widetilde{X}_\tau$. Consider two cases:

(1) $\tau \in \mathrm{Ind}(\Sigma)$. The morphism $\tilde{j}_{\pi_\tau} : \widetilde{X}_\tau \to \widetilde{X}_\tau/\Gamma_\tau = \widetilde{X}_{\tau^\Gamma}$ is G^τ-equivariant. The valuation $\tilde{j}_{\pi_\tau *}(\mathrm{val}(v)) = \mathrm{val}(\pi_\tau(v))$ is G^τ-invariant on $\widetilde{X}_\tau/\Gamma_\tau$. Thus for any $g \in G^\tau$, $g_* \tilde{j}_{\pi_\tau *}(\mathrm{val}(v)) = \tilde{j}_{\pi_\tau *} g_*(\mathrm{val}(v)) = \tilde{j}_{\pi_\sigma *}(\mathrm{val}(v))$. This means that $\mathrm{val}(v)$ and $g_*(\mathrm{val}(v))$ define the same functional on the lattice of characters M_τ^Γ and consequently M_τ (since $M_\tau^\mathbf{Q} = (M_\tau^\Gamma)^\mathbf{Q}$). This shows by Lemma 5.1.3 that $g_*(\mathrm{val}(v)) \geq \mathrm{val}(v)$ and finally by Lemma 5.1.4 we conclude $g_*(\mathrm{val}(v)) = \mathrm{val}(v)$.

(2) $\tau \in \mathrm{Dep}(\Sigma)$. By Lemma 10.3.2(2), $\widetilde{i_{/K^*}} : \widetilde{X}_{\pi(\tau_-)} \to \widetilde{X}_{\pi(\tau)}$ is a G^τ-equivariant proper birational morphism. Then the valuation $\mathrm{val}(v, \widetilde{X}_{\pi(\tau_-)}) = (\widetilde{i_{/K^*}}_*^{-1}(\mathrm{val}(v, \widetilde{X}_{\pi(\tau)}))$ is G^τ-invariant. By Lemma 10.3.5(4), $\tilde{j}_{\pi_{\tau_-}}^*(\mathrm{val}(\pi(v), \widetilde{X}_{\pi(\tau_-)})) = \mathrm{val}(\pi_{\tau_-}^{-1}\pi(v), \widetilde{X}_{\tau_-})$ is G^τ-equivariant. We get $\pi_{\tau_-}^{-1}(\pi(v)) \in \mathrm{Inv}(\tau)$. Analogously $\pi_{\tau_+}^{-1}(\pi(v)) \in \mathrm{Inv}(\tau)$. By Lemma 6.2.1 this gives $v \in \langle \pi_{\tau_-}^{-1}(\pi(v)), \pi_{\tau_+}^{-1}(\pi(v)) \rangle \subset \mathrm{Inv}(\tau)$. $\square$

Proposition 10.3.9. *Let $\sigma \in \mathrm{Ind}(\Sigma)$ be an independent face of Σ. Then*

1. *All vectors in $\overline{\mathrm{par}}(\sigma^\Gamma) \cap \mathrm{int}(\sigma^\Gamma)$ are Σ-stable.*
2. *All vectors in $\mathrm{par}(\sigma^\Gamma)$ are Σ-stable.*

Proof (1) Let $v \in \overline{\mathrm{par}}(\sigma^\Gamma) \cap \mathrm{int}(\sigma^\Gamma)$. We have to prove that it defines a G^τ-invariant valuation on any $\widetilde{X}_{\pi_\tau(\tau)}$ for $\tau \geq \sigma$.

By Lemma 10.3.6 either σ^Γ is a face of τ^Γ or $\tau \in \mathrm{Dep}(\Sigma)$ and σ^Γ is a face of $\pi_\tau(\tau_-)$ (or $\pi_\tau(\tau_+)$). In the first case σ^Γ is a is a G^τ-invariant face of τ^Γ. By Lemma 6.4.1(2), $\mathrm{val}(v)$ defines a G^τ-invariant valuation on $\widetilde{X}_{\tau^\Gamma}$. In the second case σ is a G^τ-invariant face of τ_- (or τ_+). Consequently, the closure of the orbit O_{σ^Γ} in $\widetilde{X}_{\pi_\tau(\tau_-)}$ is G_τ-invariant. Let $\widetilde{i_{/K^*}} : \widetilde{X}_{\pi_\tau(\tau_-)} \longrightarrow \widetilde{X}_{\pi_\tau(\tau)}$ be the G_τ-equivariant morphism defined by the subdivision $\pi_\tau(\tau_-)$ of $\pi_\tau(\tau)$. Then by Lemma 6.4.1(2), $\overline{\mathrm{par}}(\sigma^\Gamma) \cap \mathrm{int}(\sigma^\Gamma)$ is contained $\mathrm{Inv}(\pi_\tau(\tau))$.

(2) Let $v \in \mathrm{par}(\pi_\sigma(\sigma))$. Then $v \in \overline{\mathrm{par}}(\pi_\sigma(\tau)) \cap \mathrm{int}(\tau)$ for some Γ-indecomposable face τ of σ. We apply (1).

Lemma 10.3.10. *Let $\sigma = \langle v_1, \ldots, v_k \rangle$ be a circuit in Σ with the unique relation $\sum \alpha_i w_i = 0$, where all $\alpha_i \neq 0$ and $w_i = \mathrm{prim}(\pi_\sigma(v_i))$. Then $\mathrm{Ctr}_-(\sigma) := \sum_{\alpha_i < 0} w_i$ and $\mathrm{Ctr}_+(\sigma) := \sum_{\alpha_i > 0} w_i$ are Σ-stable.*

Proof. We have to show that for any $\tau \geq \sigma$ the valuation $\mathrm{val}(\mathrm{Ctr}_-(\sigma), \widetilde{X}_{\tau^\Gamma})$ is G_τ-invariant on $\widetilde{X}_{\tau^\Gamma}$. By Lemmas 10.2.10 and 10.3.2(1), the face $\sigma^- = \langle v_i \rangle_{\alpha_i > 0} \in \sigma_- \subset \tau_-$ corresponds to the G^τ-invariant

closed subscheme of $\widetilde{X}_{\tau_-} \subset \widetilde{X}_\tau$. Therefore the orbit closure scheme $\overline{O_{\pi_\sigma(\sigma^-)}} \subset \widetilde{X}_{\pi_\tau(\tau_-)}$ is G^τ-invariant. By Lemma 6.4.1(2), applied to the subdivision $\pi_\tau(\tau_-)$ of $\pi_\tau(\tau)$, $v := \mathrm{Ctr}_-(\sigma)$ defines a G^τ-invariant valuation on $\widetilde{X}_{\pi_\tau(\tau)}$. $\qquad\square$

Lemma 10.3.11. *Let Δ be a canonical subdivision of Σ.*

1. *If $v \in \mathrm{par}(\pi(\delta))$, where $\delta \in \Delta^\sigma$, is an independent cone, then $\mathrm{Mid}(v,\delta)$ is Σ-stable.*
2. *Let $\delta \in \Delta^\sigma$ be a circuit. Then $\mathrm{Mid}(\mathrm{Ctr}_+(\delta),\delta), \mathrm{Mid}(\mathrm{Ctr}_-(\delta),\delta)$ are Σ-stable.*

Proof (1) There is a Γ-indecomposable cone $\delta' \preceq \delta$ such that $v \in \mathrm{par}(\pi(\delta'))$. Then $\delta' \in \Delta^\sigma_{\mathrm{stab}}$. We apply Lemma 10.3.10 and Proposition 10.3.8 to $\delta' \in \Sigma' = \Delta_{\mathrm{stab}}$ to deduce that $\mathrm{Mid}(v,\delta) = \mathrm{Mid}(v,\delta')$ is Σ'-stable. By Proposition 9.2.1(3), it is Σ-stable.

(2) The circuit $\delta \in \Delta^\sigma$ is Γ-indecomposable. Thus $\delta \in \Delta_{\mathrm{stab}}$. We apply Lemma 10.3.10 and Proposition 10.3.8 to $\delta \in \Sigma' = \Delta_{\mathrm{stab}}$ to infer that $\mathrm{Mid}(\mathrm{Ctr}_+(\delta),\delta), \mathrm{Mid}(\mathrm{Ctr}_-(\delta),\delta)$ are Σ'-stable. Consequently, by Proposition 9.2.1(3) they are Σ-stable. $\qquad\square$

10.4. The π-desingularization Lemma of Morelli.

Definition 10.4.1. An independent cone $\delta \in \Delta^\sigma$ is π-*nonsingular* if $\pi_\sigma(\delta)$ is regular. A subdivision Δ is π-*nonsingular* if all independent cones in Δ^σ, where $\sigma \in \Sigma$, are π-*nonsingular*.

Definition 10.4.2. (Morelli [49], [50], [5]) Let $\pi : N^{\mathbf{Q}+} := N^{\mathbf{Q}} \oplus \mathbf{Q} \to N^{\mathbf{Q}}$ be the natural projection and $v = (\{0\} \times 1) \in N^{\mathbf{Q}}$. A fan Σ in $N^{\mathbf{Q}+}$ is called a *polyhedral cobordism* or simply a *cobordism* if

1. For any cone $\sigma \in \Sigma$ the image $\pi(\sigma)$ is strictly convex (contains no line).
2. The sets of cones

$$\Sigma_+ := \{\sigma \in \Sigma \mid \text{there exists} \quad \epsilon > 0, \text{such that } \sigma + \epsilon \cdot v \notin |\Sigma|\}$$

and

$$\Sigma_- = \{\sigma \in \Sigma \mid \quad \text{there exists} \quad \epsilon > 0, \text{such that} \sigma + \epsilon \cdot v \notin |\Sigma|\}$$

are subfans of Σ and $\pi(\Sigma_-) := \{\pi(\tau) \mid \tau \in \Sigma_-\}$ and $\pi(\Sigma_+) := \{\pi(\tau) \mid \tau \in \Sigma_+\}$ are fans in $N^{\mathbf{Q}}$.

Lemma 10.4.3. *(Morelli [49], [50], [5]) Let Σ be a simplicial cobordism in $N^{\mathbf{Q}+}$. Then there exists a simplicial cobordism Δ obtained from Σ by a sequence of star subdivisions such that Δ is π-nonsingular. Moreover, the sequence can be taken so that any independent and already π-nonsingular face of Σ remains unaffected during the process. Moreover all centers of star subdivisions are of the form (1), (2) from Lemma 10.3.11.* $\quad\square$

A simple corollary of the above lemma is

Lemma 10.4.4. *Let Σ be a simplicial strictly convex K^*-semicomplex. Then there exists a canonical subdivision Δ of Σ which is a sequence of star subdivisions such that Δ is π-nonsingular. Moreover, the sequence can be taken so that any independent and already π-nonsingular face of Σ remains unaffected during the process. Moreover all centers are of the form (1), (2) from Lemma 10.3.11 and therefore are Σ-stable.*

Lemma 10.4.5. *([72],Lemma 10, [49]) Let $w_1, \ldots, w_{k+1}$ be integral vectors in $\mathbf{Z}^k \subset \mathbf{Q}^k$ which are not contained in a proper vector subspace of $\mathbf{Q}^k$. Then*

$$\sum_{i=1}^n (-1)^i \det(w_1, \ldots, \check{w}_i, \ldots, w_{k+1}) \cdot w_i = 0$$

is the unique (up to proportionality) linear relation between $w_1, \ldots, w_{k+1}$.

Proof. Let $v := \sum_{i=1}^n (-1)^i \det(w_1, \ldots, \check{w}_i, \ldots, w_{k+1}) \cdot w_i$. Then for any $i < j$,

$$\det(w_1, \ldots, \check{w}_i, \ldots, \check{w}_j, \ldots, w_{k+1}, v) = \det(w_1, \ldots, \check{w}_i, \ldots, w_{k+1}) \cdot \det(w_1, \ldots, \check{w}_i, \ldots, \check{w}_j, \ldots, w_{k+1}, w_i) +$$
$$\det(w_1, \ldots, \check{w}_j, \ldots, w_{k+1}) \cdot \det(w_1, \ldots, \check{w}_i, \ldots, \check{w}_j, \ldots, w_{k+1}, w_j) =$$
$$(-1)^i (-1)^{k-i} \det(w_1, \ldots, \check{w}_j, \ldots, w_{k+1}) \cdot \det(w_1, \ldots, \check{w}_i, \ldots, w_{k+1}) +$$
$$(-1)^j (-1)^{k-j+1} \det(w_1, \ldots, \check{w}_i, \ldots, w_{k+1}) \cdot \det(w_1, \ldots, \check{w}_j, \ldots, w_{k+1}) = 0$$

Therefore $v \in \bigcap_{i,j} \mathrm{lin}\{w_1, \ldots, \check{w}_i, \ldots, \check{w}_j, \ldots, w_{k+1}\} = \{0\}$. $\qquad\square$

Lemma 10.4.6. *If Σ is a simplicial cobordism which is π-nonsingular and $\sigma \in \Sigma$ is a circuit then* $\mathrm{Mid}(\mathrm{Ctr}_+(\sigma), \sigma) = \mathrm{Mid}(\mathrm{Ctr}_-(\sigma), \sigma)$ *and* $\mathrm{Mid}(\mathrm{Ctr}_+(\sigma), \sigma) \cdot \Sigma$ *is π-nonsingular.*

Proof. Let $\sigma = \langle v_1, \ldots, v_k \rangle$ be a curcuit and $w_i := \mathrm{prim}(\pi(\langle v_i \rangle))$..
Then by Lemma 10.4.5, in the relation the unique relation $\sum \alpha_i w_i = 0$ all $\alpha_i = \det[w_1, \ldots, \check{w}_i, \ldots, w_k] = \pm 1$ and projections of "new independent faces" are regular since they are obtained by regular star subdivisions applied to projections of "old" independent faces. □

Proof of Lemma 10.4.4. First note that the local projections $\pi_\sigma : N_\sigma \to N_\sigma^\Gamma$ glue together and commute with subdivisions by Lemma 10.1.2 and therefore the notion of π-nonsingularity is coherent. Let $\sigma_1, \ldots, \sigma_k$ denote all the faces of Σ. We shall construct by induction on i a canonical Γ-subdivision Δ_i of Σ such that the fan $\Delta_i^{\sigma_j}$ is π-nonsingular for all $j \leq i$. Suppose Δ_i is already constructed. By Lemma 10.4.3 there exist centers $v_1, \ldots, v_k$ of the appropriate type for the cobordism $\Delta_i^{\sigma_{i+1}}$ in $N_{\sigma_{i+1}}$ such that $\langle v_k \rangle \cdot \ldots \cdot \langle v_1 \rangle \cdot \Delta_i^{\sigma_{i+1}}$ is π-nonsingular. It follows from Lemma 10.3.11 that all centers in the π-desingularization process are Σ-stable, hence by Proposition 5.4.1, $\Delta_{i+1} := \langle v_k \rangle \cdot \ldots \cdot \langle v_1 \rangle \cdot \Delta_i$ is canonical. By construction $\Delta_{i+1}^{\sigma_{i+1}}$ is π-nonsingular. Note that $\Delta_{i+1}^{\sigma_j}$ for $j < i + 1$ is obtained by star subdivisions of the π-nonsingular cobordisms $\Delta_i^{\sigma_j}$ at centers of type $\mathrm{Mid}(\mathrm{Ctr}_\pm(\delta), \delta)$ only and hence by Lemma 10.4.6 it remains π-nonsingular. □

For completeness we give a simple proof of Lemma 10.4.3. The original proof of Morelli used different centers of star subdivisions but finally it was modified by the author ([49],[50]). The complete published proof of the lemma is given in [5] ([50]). The present proof is based upon the algorithm developed in [72]. All algorithms in their final versions use the same centers of subdivisions and are closely related.

Proof of Lemma 10.4.3.

Definition 10.4.7. Let $\delta = \langle v_1, \ldots, v_k \rangle$ be a dependent cone and $w_i := \mathrm{prim}(\pi(\langle v_i \rangle))$. Then we shall call the relation $\sum_{i=1}^k r_i w_i = 0$ *normal* if it is positively normalized and $|r_i| = |\det(w_1, \ldots, \check{w}_i, \ldots, w_k)|$ for $i = 1, \ldots, k$. An independent cone σ is called an *n-cone* if $|\det(\sigma)| = n$. A dependent cone σ is called an *n-cone* if one of its independent faces is an *n*-cone and the others are *m*-cones, where $m \leq n$.

Lemma 10.4.8. *Let Σ be a simplicial coborism and $\delta = \langle v_1, \ldots, v_k \rangle$ be a maximal dependent cone in Σ. Set $w_i := \mathrm{prim}(\pi(\langle v_i \rangle))$. Let*

$$\sum_i r_i w_i = 0 \tag{0},$$

where $|r_i| = |\det(w_1, \ldots, \check{w}_i, \ldots, w_k)|$, be its normal relation (up to sign).

(1) Let $v = \mathrm{Mid}(\mathrm{Ctr}_+(\delta), \delta) \in \mathrm{int}\langle v_i \mid r_i \neq 0 \rangle$. Let $m_w \geq 1$ be the integer such that the vector $w = \frac{1}{m_w}(w_1 + \ldots + w_k)$ is primitive. Then the maximal dependent cones in $\langle v \rangle \cdot \delta$ are of the form $\delta_i = \langle v_1, \ldots, \check{v}_i, \ldots, v_k, v \rangle$.

1a. Let $r_{i_0} > 0$. Then for the maximal dependent cone $\delta_{i_0} = \langle v_1, \ldots, \check{v_{i_0}}, \ldots, v_k, v \rangle$ in $\langle v \rangle \cdot \delta$, the normal relation is given (up to sign) by

$$\sum_{r_i > 0, i \neq i_0} \frac{r_i - r_{i_0}}{m_w} w_i + \sum_{r_i < 0} \frac{r_i}{m_w} w_i + r_{i_0} w = 0. \tag{1a}$$

1b. Let $r_{i_0} < 0$. Then for the maximal dependent cone $\delta_{i_0} = \langle v_1, \ldots, \check{v}_{i_0}, \ldots, v_k, v \rangle$ in $\langle v \rangle \cdot \delta$, the normal relation is given (up to sign) by

$$\sum_{r_i < 0} -\frac{r_{i_0}}{m_w} w_i + r_{i_0} w = 0. \tag{1b}$$

(2) Let $\sigma = \langle v_i \mid i \in I \rangle$ be a codefinite face of δ. For simplicity assume that $r_i \geq 0$ for $i \in I$. Let $w = \sum_{i \in I} \alpha_i w_i \in \mathrm{par}(\pi(\sigma)) \cap \mathrm{int}(\pi(\sigma))$ and $v = \mathrm{Mid}(w, \sigma) \in \mathrm{int}(\sigma)$. Then the maximal dependent cones in $\langle v \rangle \cdot \delta$ are of the form $\delta_{i_0} = \langle v_1, \ldots, \check{v}_{i_0}, \ldots, v_k, v \rangle$, where $i_0 \in I$.

2a. Let $i_0 \in I$ and $r_{i_0} > 0$. Then for the maximal dependent cone $\delta_{i_0} = \langle v_1, \ldots, \check{v}_{i_0}, \ldots, v_k, v \rangle$ in $\langle v \rangle \cdot \delta$, the normal relation is given (up to sign) by

$$\sum_{i \in I \setminus \{i_0\}, r_i > 0} (\alpha_{i_0} r_i - \alpha_i r_{i_0}) w_i + \sum_{i \notin I, r_i > 0} \alpha_{i_0} r_i w_i + \sum_{i \in I, r_i = 0} -\alpha_i r_{i_0} w_i + \sum_{r_i < 0} \alpha_{i_0} r_i w_i + r_{i_0} w = 0. \tag{2a}$$

2b. Let $i_0 \in I$ and $r_{i_0} = 0$. For the maximal dependent cone $\delta_{i_0} = \langle v_1, \ldots, \check{v}_{i_0}, \ldots, v_k, v \rangle$, the normal relation is given (up to sign) by

$$\sum_{i \in I \setminus \{i_0\}} \alpha_{i_0} r_i w_i + 0 w_{i_0} = 0. \tag{2b}$$

Proof. It is straightforward to see that the above equalities hold. We only need to show that the relations considered are normal. For that it suffices to show that one of the coefficients is equal (up to sign) to the corresponding determinant

Comparing the coefficients of w in the above relations with the normal relations from Lemma 10.4.5 we get

1a. $\det(w_1, \dots, \breve{w}_{i_0}, \dots, w_k) = r_{i_0}$.

1b. The coefficient of w is equal to $\det(w_1, \dots, \breve{w}_{i_0}, \dots, w_k)) = r_{i_0}$,

2a. The coefficient of w is equal to $\det(w_1, \dots, \breve{w}_{i_0}, \dots, w_k) = r_{i_0}$.

2b. The coefficient of w_i, where $r_i > 0$, is equal to
$\det(w_1, \dots, \breve{w}_{i_0}, \dots, \breve{w}_i, \cdots, w_k, w) = \alpha_{i_0} \det(w_1, \dots, \breve{w}_{i_0}, \dots, \breve{w}_i, \cdots, w_k, w_{i_0}) =$
$(-1)^{k-i_0} \alpha_{i_0} \det(w_1, \dots, \breve{w}_i, \cdots, w_k, w) = (-1)^{k-i_0} \alpha_{i_0} r_i$.

$\square$

Let $\delta = \langle v_1, \dots, v_k \rangle$ be a dependent cone. Let $\sum r_i w_i = 0$ be its normal relation.

We shall consider the following kinds of dependent cones.

Type I: the cones which are *pointing down*: $\mathrm{Card}\{r_i \mid r_i < 0\} = 1$.

and the cones *pointing up*: $\mathrm{Card}\{r_i \mid r_i > 0\} = 1$.

Type $I(n,n)$: the cones of type I for which $\max\{|r_i| \mid r_i > 0\} = \max\{|r_i| \mid r_i < 0\} = n$.

Type $I(n,*)$: the cones pointing down for which $\max\{|r_i| \mid r_i > 0\} < n$ and $\max\{|r_i| \mid r_i < 0\} = n$ and the cones pointing up for which $\max\{|r_i| \mid r_i < 0\} < n$ and $\max\{|r_i| \mid r_i > 0\} = n$.

Type $I(*,n)$: the cones pointing down for which $\max\{|r_i| \mid r_i < 0\} < n$ and $\max\{|r_i| \mid r_i > 0\} = n$, and the cones pointing up for which $\max\{|r_i| \mid r_i > 0\} < n$ and $\max\{|r_i| \mid r_i < 0\} = n$.

Type II: the cones which are neither pointing down nor pointing up: $\mathrm{Card}\{r_i \mid r_i > 0\} > 1$ and $\mathrm{Card}\{r_i \mid r_i < 0\} > 1$.

Type $II(n,n)$: the cones of type II for which $\max\{|r_i| \mid r_i > 0\} = \max\{|r_i| \mid r_i < 0\} = n$.

Type $II(n,*)$: the cones of type II for which $\max\{|r_i| \mid r_i \neq 0\} = n$ but $\max\{|r_i| \mid r_i > 0\} < n$ or $\max\{|r_i| \mid r_i < 0\} < n$.

Type III: the cones which are pointing down and pointing up at the same time: $\mathrm{Card}\{r_i \mid r_i > 0\} = \mathrm{Card}\{r_i \mid r_i < 0\} = 1$.

Type $III(n,n)$: the cones of type III for which $\max\{|r_i| \mid r_i > 0\} = \max\{|r_i| \mid r_i < 0\} = n$.

Types $I(*,*), II(*,*), III(*,*)$ the cones of type I, II, III, respectively, for which $\max\{|r_i| \mid r_i \neq 0\} < n$.

Denote by n the maximum of the determinants of the projections of independent cones in the simplicial cobordism Σ. The π-desingularization algorithm consists of eliminating all dependent n-cones according to the above classification.

All normal relations below are considered up to sign.

Step 1. *Eliminating all cones of type $II(n,n)$*

Let δ be a maximal dependent cone of type $II(n,n)$ in Σ with normal relation (0) (see Lemma 10.4.8). By Lemma 10.4.8 after the star subdivision at $\mathrm{Mid}(\mathrm{Ctr}_+, \delta)$ all new dependent $\delta' = \delta_{i_0}$ cones have normal relations of type (1a) if $r_{i_0} > 0$ or (1b) if $r_{i_0} > 0$. If $r_{i_0} > 0$ in the relation (1a) the absolute values of the coefficients satisfy $\frac{1}{m_w}|r_i - r_{i_0}| < n$, the absolute values of the other coefficients can be n only for some $r_i < 0$ and r_{i_0}. If $r_{i_0} < n$ then there is no positive coefficient in (1a) which is equal to n. Hence δ' can be of type $I(n,*), I(*,n), I(*,*), II(n,*)$ or $II(*,n)$. If $r_{i_0} = n$ then there is only one positive coefficient (of w) in (1a). The other coefficients are $\frac{r_i - r_{i_0}}{m_w} = \frac{r_i - n}{m_w} \leq 0$ and $r_i < 0$. Thus δ' is of the form $I(n,*)$ or $I(n,n)$.

If $\delta' = \delta_{i_0}$ for $r_{i_0} < 0$ then the normal relation is of the form (1b). If $|r_{i_0}| = n$ then δ' is of the form $I(n,n)$. If $|r_{i_0}| < n$ then δ' is of the form $I(*,*)$.

By taking the star subdivision we eliminate dependent cones in $\mathrm{Star}(\delta_+, \Sigma)$, with the normal relation proportional to (0). All the dependent n-cones we create are either of type I or $II(n,*)$.

Step 2. *Eliminating all cones of type $I(n,n)$*. Let δ be a maximal dependent cone of type $I(n,n)$. Without loss of generality we may assume that δ is pointing down (one negative ray). Then δ^+ is of codimension 1 and $\det(\pi(\delta^+)) = n$.

Find $v_+ \in \operatorname{par}(\pi(\delta^+))$. Then $v_+ \in \operatorname{int}(\pi(\sigma_v))$, where σ_v is a face of δ^+. Thus σ_v is independent.

Step 2a. *Making σ_v codefinite with respect to all dependent cones in* $\operatorname{Star}(\sigma_v, \Sigma)$.

By applying certain star subdivisions we eliminate all cones for which σ_v is not codefinite. Let $\delta' \in \operatorname{Star}(\sigma_v, \Sigma)$ be any dependent cone for which σ_v is not codefinite. Note that δ' is not of type III since in this case the circuit consists of one positive and one negative ray and the noncodefinite face σ_v containing positive and negative rays would contain a circuit, which is impossible for an independent cone. Therefore δ' is either of type I or II.

Suppose first that $\delta' = \langle v_1, \dots, v_k \rangle$ is of type $I(n,n)$ or $I(*,n)$. Without loss of generality we may assume that there is only one coefficient $r_{i_1} > 0$ (equal to n) in the normal relation (0) from Lemma 10.4.8(1).

Since σ_v is not codefinite with respect to δ', we have $v_{i_1} \in \sigma_v$. Note that $\operatorname{Ctr}_+(\delta') = \operatorname{prim}(\pi(\langle v_{i_1} \rangle))$ and for $v := \operatorname{Mid}(\operatorname{Ctr}_+(\delta'), \delta')$ we have $\operatorname{prim}(\pi(\langle v_{i_1} \rangle)) = \operatorname{prim}(\pi(\langle v_{i_1} \rangle))$. Let $w_i := \operatorname{prim}(\pi(\langle v_{i_1} \rangle))$ and $w := \operatorname{prim}(\pi(\langle v \rangle))$. Then $w = w_{i_1}$. After the star subdivision of δ' at $\langle v \rangle$ by Lemma 10.4.8(1a), we create a dependent cone δ'_{i_1} in $\langle v \rangle \cdot \delta'$ with the normal relation obtained from the relation (0) by replacing w with w_{i_1}. Since σ_v contains only negative rays of δ'_{i_1} it is codefinite. We also create dependent cones δ'_{i_0}, where $i_0 \neq i_1$, of type III ($III(n,n)$ or $III(*,*)$) with the normal relation $r_{i_0} w - r_{i_0} w_{i_1} = 0$ (case 1b). The face σ_v is codefinite with respect to δ'_{i_0}. Thus after blow-up we create one cone of the type $I(n,n)$ and cones of type $III(n,n)$ or $III(*,*)$. Since at the same time we eliminate the cone δ', the number of maximal cones of type $I(n,n)$ remains the same. After the star subdivision σ_v is codefinite with respect to all new dependent cones.

Suppose δ' is of type $I(n,*)$, $I(*,*)$ $II(n,*)$ or $II(*,*)$. Without loss of generality we may assume that in the normal relation (0) for δ' (see Lemma 10.4.8(1)):

$\max\{r_i \mid r_i > 0\} \geq \max\{|r_i| \mid r_i < 0\}$.

In particular $\max\{|r_i| \mid r_i < 0\} < n$.

We shall assign to such a cone δ' for which σ_v is not codefinite, the invariant $\operatorname{inv}(\delta') := (\max\{r_i \mid r_i > 0\}, \max\{|r_i| \mid r_i > 0\})$, ordered lexicogragraphically.

By Lemma 10.4.8(1)), after the star subdivision at $\operatorname{Mid}(\operatorname{Ctr}_+(\delta'), \delta')$ all new dependent cones $\delta'' = \delta'_{i_0}$ have normal relations of type (1a) if $r_{i_0} > 0$ or (1b) if $r_{i_0} < 0$.

If $r_{i_0} = \max\{r_i \mid r_i > 0\}$ then in the relation (1a) there is only one positive coefficient of w which is equal to r_{i_0}. All other coefficients: $\frac{r_i - r_{i_0}}{m_w}$ and $r_i < 0$ are nonpositive with the absolute values $< n$. Thus δ'_{i_0} is of the type $I(n,*)$ or $I(*,*)$. The face σ_v is codefinite with respect to δ'_{i_0}.

If $0 < r_{i_0} < \max\{r_i \mid r_i > 0\}$ then the absolute values of the coefficients $\frac{r_i - r_{i_0}}{m_w}$, r_{i_0} and $r_i \leq 0$, are $< n$. The cone δ'_{i_0} is of type $I(*,*)$, $II(*,*)$, $III(*,*)$ and $\operatorname{inv}(\delta'_{i_0}) < \operatorname{inv}(\delta')$.

If $r_{i_0} < 0$ then the absolute values of all coefficients in the relation (1b) are equal to $|r_{i_0}| < n$ or $|\frac{r_{i_0}}{m_w}| < n$ and δ'_{i_0} is a dependent cone of type $I(*,*)$ for which σ_v is not codefinite.

We repeat the procedure for all cones δ'' for which σ_v is not codefinite. The procedure terminates since for all new dependent cones for which σ_v is not codefinite the invarinat $\operatorname{inv}(\delta')$ drops.

Step 2b. *Eliminating δ*. Apply the star subdivision at $\langle v \rangle$ to the resulting cobordism. We eliminate all cones in $\operatorname{Star}(\sigma_v)$. In particular $\delta \in \operatorname{Star}(\sigma_v)$ will be eliminated. By Step 2(a) σ_v is codefinite with respect to all dependent cones in $\operatorname{Star}(\sigma_v)$. All such cones δ' can be of the type $I(n,n)$, $I(*,n)$ $I(n,*)$, $I(*,*)$, $II(n,*)$, $II(*,*)$, $III(n,n)$, $III(*,*)$.

By Lemma 10.4.8(2) all new dependent cones after the star subdivision have normal relations of the form (2a) and (2b).

Since all the coefficients in normal relations of the form (2b) are $< n$, the corresponding dependent cones are of the form $I(*,*)$, $II(*,*)$, $III(*,*)$.

In the normal relation of the form (2a) for δ'_{i_0}, the absolute values of the coefficients of w_i, for $i \neq i_1$, are $< n$. The coefficient r_{i_0} of w is the only one which can be equal to n. Therefore all new dependent cones we create are of the type $I(n,*)$, $I(*,*)$, $II(n,*)$, $II(*,*)$, $III(*,*)$.

Step 3. *Eliminating all cones of type $I(*,n)$ and $II(n,*)$*.

Let $\delta = \langle v_1, \ldots, v_k \rangle$ be a cone of the type $I(*, n)$ or $II(n, *)$.

We may assume that the positive coefficients r_i in the normal relation are $\leq n$ and the absolute values of all negative coefficients are $< n$. Then δ^+ contains at least two positive rays.

By Lemma 10.4.8(1) after the star subdivision at $\mathrm{Mid}(\mathrm{Ctr}_+(\delta), \delta)$ we create new dependent n-cones δ_{i_0} only for $r_{i_0} = n$ in case (1a). Then by the same reasoning as in Step1, δ_{i_0} is of type $I(n, *)$.

Step 4. *Eliminating all cones of type* $III(n, n)$.

Let δ be a cone of type $III(n, n)$. Then $\pi(\delta^+) = \pi(\delta^-)$. Find $w \in \mathrm{par}(\pi(\delta^+))$ and let $v := \mathrm{Mid}(w, \delta^+)$. Then v is in the relative interior of a face σ_v of δ^+.

Step 4a. *Making* σ_v *codefinite with respect to all dependent cones in* $\mathrm{Star}(\sigma_v, \Sigma)$.

Let $\delta' \in \mathrm{Star}(\sigma_v, \Sigma)$ be any dependent cone for which σ_v is not codefinite. As in Step 2, δ' can be only of type I or II. Then δ' is of type $I(n, *)$, $I(*, *)$ or $II(*, *)$.

Suppose δ' is of type $I(n, *)$. Without loss of generality we may assume that in the normal relation for δ' there is only one positive coefficient r_{i_1} equal to n and $|r_i| < n$ for all negative coefficients r_i.

Since σ_v is not codefinite with respect to δ', we have $v_{i_1} \in \sigma_v$. Note that $\mathrm{Ctr}_+(\delta') = \langle v_{i_1} \rangle$ and for $v := \mathrm{Mid}(\mathrm{Ctr}_+(\delta'), \delta')$ we have $w = w_{i_1}$. After the star subdivision at $\langle v' \rangle$ by Lemma 10.4.8(1) we create one dependent cone δ'_{i_1} with the normal relation obtained from the normal relation for δ' by replacing v with v_{i_1}, and some dependent cones δ'_{i_0}, for $i_0 \neq i_1$, which are of type $III(*, *)$ with the normal relation $r_{i_0} w = r_{i_0} w_{i_1}$. After the star subdivision σ_v is not codefinite with respect to any new dependent cones.

Suppose δ' is of type $I(*, *)$ or $II(*, *)$. Without loss of generality we may assume that in the normal relation (0) for δ' (see Lemma 10.4.8(1)):

$\max\{r_i \mid r_i > 0\} \geq \max\{|r_i| \mid r_i < 0\}$.

In particular $\max\{|r_i| \mid r_i < 0\} < n$.

We shall assign to such a cone δ' for which σ_v is not codefinite, the invariant $\mathrm{inv}(\delta') := (\max\{r_i \mid r_i > 0\}, \max\{|r_i| \mid r_i > 0\})$, ordered lexicogragraphically.

By Lemma 10.4.8(1)), after the star subdivision at $\mathrm{Mid}(\mathrm{Ctr}_+(\delta'), \delta')$ all new dependent cones $\delta'' = \delta'_{i_0}$ have normal relations of type (1a) if $r_{i_0} > 0$ or (1b) if $r_{i_0} < 0$.

If $r_{i_0} = \max\{r_i \mid r_i > 0\}$ then in the relation (1a) there is only one positive coefficient of w which is equal to $r_{i_0} < n$. All other coefficients: $\frac{r_i - r_{i_0}}{m_w}$ and $r_i < 0$ are nonpositive with the absolute values $< n$. Thus δ'_{i_0} is of the type $I(*, *)$. The face σ_v is codefinite with respect to δ'_{i_0}.

If $0 < r_{i_0} < \max\{r_i \mid r_i > 0\}$ then the absolute values of the coefficients $\frac{r_i - r_{i_0}}{m_w}$, r_{i_0} and $r_i \leq 0$, are $< n$. The cone δ'_{i_0} is of type $I(*, *)$, $II(*, *)$, $III(*, *)$ and $\mathrm{inv}(\delta'_{i_0}) < \mathrm{inv}(\delta')$.

If $r_{i_0} < 0$ then the absolute values of all coefficients in the relation (1b) are equal to $|r_{i_0}| < n$ or $|\frac{r_{i_0}}{m_w}| < n$ and δ'_{i_0} is a dependent cone of type $I(*, *)$ for which σ_v is not codefinite.

We repeat the procedure for all cones δ'' for which σ_v is not codefinite. The procedure terminates since for all new dependent cones for which σ_v is not codefinite the invarinat $\mathrm{inv}(\delta')$ drops.

Step 4b. *Eliminating* δ. Apply the star subdivision at $\langle v \rangle$ to the resulting cobordism. We eliminate all cones in $\mathrm{Star}(\sigma_v)$. In particular $\delta \in \mathrm{Star}(\sigma_v)$ will be eliminated. By Step 4(a) σ_v is codefinite with respect to all dependent cones in $\mathrm{Star}(\sigma_v)$. All such cones δ' can be of the type $I(n, *)$, $I(*, *)$, $II(*, *)$, $III(n, n)$, $III(*, *)$.

By Lemma 10.4.8(2) all new dependent cones after the star subdivision have normal relations of the form (2a) and (2b).

Since all the coefficients in normal relations of the form (2b) are $< n$, the corresponding dependent cones we create are of the form $I(*, *)$, $II(*, *)$, $III(*, *)$.

If δ' is of type $I(*, *)$, $II(*, *)$, $III(*, *)$ then all the coefficients in normal relations of the form (2a) are $< n$ and the corresponding dependent cones we create are of the form $I(*, *)$, $II(*, *)$, $III(*, *)$.

If δ' is of type $I(n, *)$ then we can assume that σ_v is a face of δ'^+. If δ' is pointing down then all positive coefficients r_i in the normal relation for δ' are $< n$. In the normal relation (2a) for δ'_{i_0}, where $r_{i_0} > 0$, all coefficients are $< n$. If δ' is pointing up then there is only one positive coefficient r_{i_1}. In the normal relation (2a) for δ'_{i_1}, the first two sums disappear and there is only one positive coefficient of w equal to $r_{i_0} \leq n$. By

applying the star subdivision at $\langle v \rangle$ to δ' we create one maximal dependnent cone of type $I(n, *)$. At the same time we eliminate the cone δ' of type $I(n, *)$.

If δ' is of type $III(n, n)$ (for instance if $\delta' = \delta$) then there is one positive coefficient $r_{i_1} = n$ and one negative coefficient $r_{i_2} = -n$ in the normal relation for δ'.

After the star subdivision we create only one maximal dependent cone δ'_{i_0} for $i_0 = i_1$ having the normal relation of the form (2a). In the relation (2a) the first two sums disappear and there is only one positive coefficient of w equal to $r_{i_0} \leq n$. The absolute values of the other negative coefficients are $< n$. Thus we create one maximal dependent cone of type $I(n, *)$.

Step 5. *Eliminating all dependent cones of type $I(n, *)$ and all n-cones which are not faces of dependent cones.*

Let δ be a cone of type $I(n, *)$ or an n-cone which is not a face of a dependent cone.

In the first case without loss of generality we may assume that in the normal relation there is one negative coefficient equal to $-n$ and all positive coefficients are $< n$ (δ is pointing down).

Then $\pi(\delta^+) = \pi(\delta)$. Find $w \in \mathrm{par}(\pi(\delta^+))$ and let $v = \mathrm{Mid}(w, (\delta_+))$. Then $v \in \mathrm{int}(\sigma_v)$, where σ_v is a face of δ_+.

Step 5a. *Making σ_v codefinite with respect to all dependent cones in $\mathrm{Star}(\sigma_v, \Sigma)$.* We repeat word by word the procedure in Step 4a.

Step 5b. *Eliminating δ.* Apply the star subdivision at $\langle v \rangle$ to the resulting cobordism. We eliminate all cones in $\mathrm{Star}(\sigma_v)$. By Step 5(a) σ_v is codefinite with respect to all dependent cones in $\mathrm{Star}(\sigma_v)$. All such cones δ' can be of the type $I(n, *)$, $I(*, *)$, $II(*, *)$, $III(*, *)$. We repeat word by word the procedure in Step 5a excluding the last case $III(n, n)$. By applying the star subdivision at $\langle v \rangle$ we decrease the number of maximal cones of type $I(n, *)$ (or independent n-cones). All new dependent cones we create are of type $I(n, *)$, $I(*, *)$, $II(*, *)$.

$\square$

11. BIRATIONAL COBORDISMS

11.1. **Definition of a birational cobordism.**

Definition 11.1.1. ([73]): Let X_1 and X_2 be two birationally equivalent normal varieties. By a *birational cobordism* or simply a *cobordism* $B := B(X_1, X_2)$ between them we understand a normal variety B with an algebraic action of K^* such that the sets

$$B_- := \{x \in B \mid \lim_{\mathbf{t} \to 0} \mathbf{t}x \text{ does not exist}\} \quad \text{and}$$
$$B_+ := \{x \in B \mid \lim_{\mathbf{t} \to \infty} \mathbf{t}x \text{ does not exist}\}$$

are nonempty and open and there exist geometric quotients B_-/K^* and B_+/K^* such that $B_+/K^* \simeq X_1$ and $B_-/K^* \simeq X_2$ and the birational map $X_1 - \to X_2$ is given by the above isomorphisms and the open embeddings $B_+ \cap B_-/K^*$ into B_+/K^* and B_-/K^* respectively.

11.2. **Collapsibility.** Let X be a variety with an action of K^*. Let $F \subset X$ be a set consisting of fixed points. Then we define

$$F^+(X) = F^+ = \{x \in X \mid \lim_{\mathbf{t} \to 0} \mathbf{t}x \in F\}, \ F^-(X) = F^- = \{x \in X \mid \lim_{\mathbf{t} \to \infty} \mathbf{t}x \in F\}.$$

Definition 11.2.1. ([73]). Let X be a cobordism or any variety with a K^*-action.

1. We say that a connected component F of the fixed point set is an *immediate predecessor* of a component F' if there exists a nonfixed point x such that $\lim_{\mathbf{t} \to 0} \mathbf{t}x \in F$ and $\lim_{\mathbf{t} \to \infty} \mathbf{t}x \in F'$.
2. We say that F *precedes* F' and write $F < F'$ if there exists a sequence of connected fixed point set components $F_0 = F, F_1, \ldots, F_l = F'$ such that F_{i-1} is an immediate predecessor of F_i (see [7] , Def. 1. 1).
3. We call a cobordism (or a variety with K^*-action) *collapsible* (see also Morelli [49]) if the relation $<$ on its set of connected components of the fixed point set is an order. (Here an order is just required to be transitive.)

Lemma 11.2.2. *([73]). A projective cobordism is collapsible.* ☐

Lemma 11.2.3. *Let B be a collapsible variety and I be a K^*-invariant sheaf of ideals. Then the blow-up $B' := \mathrm{bl}_I(B)$ is also collapsible.*

Proof. Set $\pi : B' \to B$. For any connected fixed point component F' on B' by abuse of notation denote by $\pi(F')$ the fixed point component containing the image $\pi(F')$. Then for any F_0' and F_1' on B' the relation $F_0' < F_1'$ implies that either $\pi(F_0') < \pi(F_1')$ or $\pi(F_0') = \pi(F_1')$. Consider the latter case. Let $x \in F'$ and $u_1, \ldots, u_k$ be semiinvariant generators. Then at least one of the functions $\pi^*(u_i)$, say $\pi^*(u_{i_0})$, generates $\pi^{-1}(I) \cdot \mathcal{O}_{B'}$ at x. We let $\mathrm{ord}(F)$ denote the weight of $\pi^*(u_{i_0})$ with respect to the K^*-action. This definition does not depend on the choice of the semiinvariant generator. For any two generators $\pi^*(u_{i_0})$ and $\pi^*(u_{i_1})$ their quotient is a semiinvariant function which is invertible at the fixed point $x \in B'$, which implies that it is invariant function. This definition is locally constant, which means that it does not depend on the choice of $x \in X$. Moreover the functions $\pi^*(u_1), \ldots, \pi^*(u_k)$ are sections generating $\mathcal{O}_{\pi^{-1}(U)}(-D)$, where D is the exceptional divisor of π. This gives a K^*-equivariant morphism $\phi : \pi^{-1}(U) \to \mathbf{P}^k$ into a projective space with semiinvariant coordinates $x_1, \ldots x_k$ having the same weights as $u_1, \ldots, u_k$, say $a_1, \ldots, a_k$. We can assume that $a_1 \leq \ldots \leq a_k$. Then the function $\mathrm{ord}(F) = \min\{a_i \mid x_i(\phi(F)) \neq 0\}$ coincides with the one induced on $\mathbf{P}^k$, and $F < F'$ implies $\mathrm{ord}(F) < \mathrm{ord}(F')$. Define $F \prec F'$ if either $\pi(F) < \pi(F')$, or $\pi(F) = \pi(F')$ and $\mathrm{ord}(F) \leq \mathrm{ord}(F')$. Since $F < F'$ implies $F \prec F'$ it follows that $\prec$ is an order on the fixed point components on B'. ☐

11.3. Decomposition of a birational cobordism.

Definition 11.3.1. *([73]). Let B be a collapsible cobordism and F_0 be a minimal component. By an elementary collapse with respect to F_0 we mean the cobordism $B^{F_0} := B \setminus F_0^-$ (see section 10.3). By an elementary cobordism with respect to F_0 we mean the cobordism $B_{F_0} := B \setminus \bigcup_{F \neq F_0} F^+$.*

Proposition 11.3.2. *([73]). Let F_0 be a minimal component of the fixed point set in a collapsible cobordism B. Then the elementary collapse B^{F_0} with respect to F_0 is again a collapsible cobordism, in particular it satisfies:*

1. *$B_+^{F_0} = B_+$ is an open subset of B,*
2. *F_0^- is a closed subset of B and equivalently B^{F_0} is an open subset of B.*
3. *$B_-^{F_0}$ is an open subset of B^{F_0} and $B_-^{F_0} = B^{F_0} \setminus \bigcup_{F \neq F_0} F^+$.*
4. *The elementary cobordism B_{F_0} is an open subset of B such that*

$$(B_{F_0})_- = B_{F_0} \setminus F_0^+ = B_-$$
$$(B_{F_0})_+ = B_{F_0} \setminus F_0^- = B_-^{F_0}.$$

5. *There exist good and respectively geometric quotients $B_{F_0}//K^*$ and $B^{F_0}_-/K^*$ and moreover the natural embeddings $i_- : B_- \subset B_{F_0}$ and $i_+ : B_-^{F_0} \subset B_{F_0}$ induce proper morphisms $i_{-/K^*} : B_-^{F_0}/K^* \to B_{F_0}//K^*$ and $i_{+/K^*} : B_-/K^* \to B_{F_0}//K^*$.* ☐

We can extend the order $<$ from Definition 11.2.1 to a total order.

As a corollary from the above we obtain:

Proposition 11.3.3. *Let $F_1 < \ldots < F_k$ denote the connected fixed point components of a cobordism B.*

Set $B_i := B \setminus (\bigcup_{F_j < F_i} F_j^- \cup \bigcup_{F_j > F_i} F_j^+)$.

Then

1. *$(B_1)_- = B_-$, $(B_k)_+ = B_+$.*
2. *$(B_{i+1})_- = (B_i)_+$.*
3. *There is a factorization*

$$B_-/K^* = (B_1)_-/K^* - \to (B_1)_+/K^* = (B_2)_-/K^* - \to \ldots (B_{k-1})_+/K^* = (B_k)_-/K^* - \to (B_k)_+/K^* = B_+/K^*.$$

☐

11.4. **Existence of a smooth birational cobordism.**

Proposition 11.4.1. *(see also* [73]*) Let $\phi : Y \to X$ be a birational projective morphism between smooth complete varieties over an algebraically closed field of characteristic 0. Let $U \subset X, Y$ be an open subset , where ϕ is an isomorphism.*

1. *There exists a smooth collapsible cobordism $B = B(X,Y)$ between X and Y such that $B \supset U \times K^*$*
2. *If $D_X := X \setminus U$ and $D_Y := Y \setminus U$ are divisors with normal crossings then there exists a cobordism $\tilde{B}$ between $\tilde{X}$ and $\tilde{Y}$ such that*
   - *$\tilde{X}$ and $\tilde{Y}$ are obtained from X and Y by a sequence of blow-ups at centers which have normal crossings with components of the total transforms of D_X and D_Y respectively.*
   - *$U \times K^* \subset \tilde{B}$ and $\tilde{B} \setminus (U \times K^*)$ is a divisor with normal crossings.*

Proof. (1) If X and Y are projective the construction of B is given in Proposition 2 of [73]; in general we use the construction of D.Abramovich (Remark after Proposition 2 in [73]). B is an open subset of $\overline{B}$ where $\overline{B}$ is obtained from $X \times \mathbf{P}^1$ by a sequence of blow-ups at ideals, hence by a single blow-up of an ideal. By Lemma 11.2.3, $\overline{B}$ is collapsible. Consequently B is a collapsible cobordism.

(2) The sets B_- and B_+ are locally trivial K^*-bundles with projections $\pi_- : B_- \to B_-/K^* \simeq X$ and $\pi_+ : B_+ \to B_+/K^* \simeq Y$. Let $Z := B \setminus (U \times K^*)$. Then $Z \cap B_-$ and $Z \cap B_+$ are normal crossing divisors and $\pi_-(Z \cap B_-) = D_X$ and $\pi_+(Z \cap B_+) = D_Y$. Let $f : \tilde{B} \to B$ be a canonical principalization of I_Z (see Hironaka [29], Villamayor [71] and Bierstone-Milman [8]). Let $f_+ : f^{-1}(B_+) \to B_+$ (resp.$f_- : f^{-1}(B_-) \to B_-$) be its restriction. By functoriality f_+ (resp. f_-) is a canonical principalization of B_+ (resp. B_-) which commutes with a canonical principalization $\tilde{Y}$ of I_{D_Y} on Y (resp. $\tilde{X}$ of I_{D_X} on D_X) In particular all centers are K^*-invariant of the form $\pi_+^{-1}(C)$ (resp. $\pi_-^{-1}(C)$), where C have normal crossings with components of the total transform of D_Y (resp. D_X). $\qquad\square$

12. Weak factorization theorem

12.1. **Construction of a π-regular cobordism.**

Definition 12.1.1.   1. We say that a cobordism B is *π-regular* if for any x in B which is not a fixed point of the K^*-action in B there exists an affine invariant neighborhood U without fixed points such that U/K^* is smooth.
2. We call a cobordism B *K^*-stratified* if it is a K^*-stratified toroidal variety.

Proposition 12.1.2. *Let (B, S) be a smooth stratified cobordism between smooth varieties X and X'. There exists a K^*-toroidal modification $\tilde{B}$ of B obtained as a sequence of blow-ups of stable valuations such that $\tilde{B}$ is a π-regular K^*-stratified cobordism between X and X'. Moreover if $U \subset X$ is an open affine invariant fixed point free subset such that U/K^* is smooth then all centers of blow-ups are disjoint from U.*

Proof. By Lemma 4.8.3 we can associate with (B, S) a K^*-semicomplex Σ. Since for any fixed point component F in B the sets F^+ and F^- are not dense it follows that Σ_S is strictly convex. By Lemma 10.4.4 there is a K^*-canonical subdivision Δ of Σ which is π-nonsingular and does not affect any π-nonsingular cones in Σ. By Theorem 6.6.1, Δ corresponds to a K^*-stratified cobordism between X and Y. $\qquad\square$

12.2. **Factorization determined by a π-regular cobordism.**

Proposition 12.2.1. *Let (B, R) be a π-regular collapsible K^*-stratified cobordism between smooth stratified toroidal varieties $(X, S_X) = (B_-/K^*, (R \cap B_-)/K^*)$, $(Y, S_Y) = (B_+/K^*, (R \cap B_+)/K^*)$. Let B_i be elementary cobordisms as in Proposition 11.3.3. Then there exists a sequence of smooth stratified toroidal varieties $(X_i, S_i) := ((B_i)_-/K^*, (R \cap (B_i)_-)/K^*)$ and a factorization*

$$
\begin{array}{ccccccccccc}
& & Y_0 & & & & Y_1 & & & Y_{n-1} & \\
& g_0 \nearrow & & \searrow f_0 & g_1 \nearrow & & \searrow f_1 & & g_{n-1} \nearrow & & \searrow f_{n-1} \\
X = X_0 & & \longrightarrow & & X_1 & & \longrightarrow & & X_2 \ \ \dots \ \ X_{n-1} & \longrightarrow & X_n = Y
\end{array} ,
$$

where each X_i is a smooth variety and $f_i : Y_i \to X_{i+1}$ and $g_i : Y_i \to X_i$ for $i = 0, \dots, n-1$ are blow-ups at smooth centers which have normal crossings with the closures of strata in S_i (resp. S_{i+1}).

Proof. Follows from the following Lemma 12.2.2 and Proposition 11.3.3. $\qquad\square$

Lemma 12.2.2. *Let (B, R) be a π-regular elementary K^*-stratified cobordism. Then $X = B_-/K^*$ and $Y = B_+/K^*$ are smooth stratified toroidal varieties with stratifications S_X and S_Y induced by $R \cap B_+$ and $R \cap B_-$ and there exists a smooth variety Z, obtained by a blow-up at a smooth center C_X (resp. C_Y), having only normal crossings with closures of strata in S_X (resp. S_X).*

Proof. For any X let $\widetilde{X}$ denote its normalization.

The fixed point component is equal to the closure $\overline{\mathrm{strat}_B(\sigma)}$ of the maximal stratum , corresponding to a circuit σ.

Consider the commutative diagram

$$
\begin{array}{ccc}
B_+/K^* & & B_-/K^* \\
& \searrow \qquad \swarrow & \\
& B//K^* &
\end{array}
$$

This diagram can be completed to

$$
Z := (B_+/K^* \times_{B//K^*} B_-/K^*)\widetilde{}
$$
$$
\psi \downarrow
$$
$$
B_+/K^* \times_{B//K^*} B_-/K^*
$$

$$
\begin{array}{ccccc}
& \swarrow & & \searrow & \\
B_+/K^* & & & & B_-/K^* \qquad (*) \\
& \searrow & & \swarrow & \\
& & B//K^* & &
\end{array}
$$

where the morphism ψ is given by the normalization.

For any $x \in \overline{\mathrm{strat}(\sigma)}$ find $\tau \geq \sigma$ such that $x \in \mathrm{strat}(\tau)$. Let $\phi_\tau : U_\tau \to X_\tau$ be the relevant chart.

The above diagram defines locally a diagram

$$
((U_\tau)_+/K^* \times_{U_\tau/K^*} (U_\tau) - /K^*)\widetilde{}
$$
$$
\downarrow
$$
$$
(U_\tau)_+/K^* \times_{U_\tau/K^*} (U_\tau)_-/K^*
$$

$$
\begin{array}{ccccc}
& \swarrow & & \searrow & \\
(U_\tau)_+/K^* & & & & (U_\tau)_-/K^* \\
& \searrow & & \swarrow & \\
& & U_\tau//K^* & &
\end{array}
$$

This diagram is a pull-back via a smooth morphism $\phi_{\tau/K^*} : U_\tau//K^* \to X_\tau/K^*$ of the diagram of toric varieties

$$
X_\Sigma := ((X_\tau)_-/K^* \times_{X_\tau/K^*} (X_\tau)_+/K^*)\widetilde{}
$$
$$
\downarrow
$$
$$
(X_\tau)_-/K^* \times_{X_\tau/K^*} (X_\tau)_+/K^*
$$

$$
\begin{array}{ccccc}
& \swarrow & & \searrow & \\
X_{\Sigma_1} := (X_\tau)_-/K^* & & & & X_{\Sigma_2} := (X_\tau)_+/K^* \\
& \searrow & & \swarrow & \\
& & X_{\tau^\Gamma} = X_\tau//K^* & &
\end{array}
$$

It follows from the universal property of the fiber product that X_Σ is a normal toric variety whose fan consists of the cones $\{\tau_1 \cap \tau_2 \mid \tau_1 \in \Sigma_1, \tau_2 \in \Sigma_2\}$.

The cone τ is π-nonsingular and by Lemma 10.4.5 we can write the only dependence relation of $\tau^\Gamma = \langle v_1, \ldots, v_k, w_1, \ldots, w_m, q_1, \ldots, q_l \rangle$, where $\langle v_1, \ldots, v_k, w_1, \ldots, w_m \rangle$ is a curcuit, as follows:

$$
e := v_1 + \ldots + v_k = w_1 + \ldots + w_m.
$$

Finally, $\Sigma = \langle e \rangle \cdot \Sigma_1$ and $\Sigma = \langle e \rangle \cdot \Sigma_2$ are regular star subdivisions.

We have shown that the morphisms $Z \to X$ and $Z \to Y$ defined by the diagram $(*)$ are blow-ups at smooth centers. These centers have normal crossings with closures of strata since locally the centers and the closures of strata are determined by the closures of orbits on a smooth toric variety. $\qquad \square$

12.3. **Nagata's factorization.**

Lemma 12.3.1. *(Nagata) Let $X \supset U \subset Y$ be two complete varieties over an algebraically closed field, containing an open subset U. Then there exists a variety Z which is simultaneously a blow-up of a coherent sheaf of ideals $\mathcal{I}_X$ on X and a blow-up of a coherent sheaf of ideals $\mathcal{I}_Y$ on Y. Moreover the supports of $\mathcal{I}_X$ and $\mathcal{I}_Y$ are disjoint from U*

The above lemma is a refinement of the original lemma of Nagata

Definition 12.3.2. (Nagata [54]) Let $X_i \supset U$, for $i = 1, \ldots, n$ be complete varieties over an algebraically closed field, containing an open subset U. By the *join* $X_1 * \ldots * X_k$ we mean the closure of the diagonal $\Delta(U) \subset X_1 \times \ldots \times X_k$.

Lemma 12.3.3. *(Nagata [54]) Let $X \supset U \subset Y$ be two complete varieties over an algebraically closed field, containing an open subset U. Then there exist coherent sheaves of ideals $\mathcal{I}_1, \ldots, \mathcal{I}_k$ on X with supports disjoint from U and blow-ups $X_1 = \mathrm{bl}_{\mathcal{I}_1} X, \ldots, X_k = \mathrm{bl}_{\mathcal{I}_k} X$ such that the join $X_1 * \ldots * X_k$ dominates Y.* $\qquad\square$

Proof of Lemma 12.3.1 Let X_i be as in the assertion of Lemma 12.3.3. Note that $X_1 \times \ldots \times X_k$ is projective over $X \times \ldots \times X$. This implies that $X_1 * \ldots * X_k$ is projective over $\Delta(X) \simeq X$. Thus there exists a sheaf of ideals $\mathcal{J}$ on X such that $X_1 * \ldots * X_k \to X$ is the blow-up at $\mathcal{J}$. By Nagata and the above find a blow-up X' of X at $\mathcal{J}$ such that X' dominates Y. Denote the blow-up by $\phi : X' \to Y$. Analogously find a blow-up Z of Y at $\mathcal{I}_Y$ such that Z dominates X'. Thus $Z \to X'$ is a blow-up at $\phi^{-1}(\mathcal{I}_Y)$ and $Z \to X$ is a composite of blow-ups and consequently a single blow-up at some sheaf of ideals $\mathcal{I}_X$. $\qquad\square$

12.4. **Proof of the Weak Factorization Theorem. Step 1** By Nagata's Lemma 12.3.1 we find a Z' obtained from X and Y respectively by blow-ups at centers disjoint from U. If X and Y are projective we can take Z' to be the graph of $\phi : X - \to Y$. Let Z'' be a resolution of singularities of Z' and Z be a canonical principalization of the ideal of the set $Z'' \setminus U$ (this is needed for part (2) of the theorem only). Then $Z \setminus U$ is a divisor with normal crossings. It suffices to prove the theorem for the projective morphisms $Z \to X$ (or $Z \to Y$).

Step 2 By Proposition 11.4.1 there exists a smooth collapsible cobordism $B = B(Z, X)$. If $D_X := X \setminus U$ and $D_Z := Z \setminus U$ are divisors with normal crossings then there exists a cobordism $\tilde{B}$ between $\tilde{X}$ and $\tilde{Y}$ as in the proposition such that $\tilde{B} \setminus (U \times K^*)$ is a normal crossings divisor. For simplicity denote $\tilde{X}, \tilde{Z}, \tilde{B}$ by X, Z, B respectively.

Step 3 By Lemma 4.2.1 there exists a K^*-invariant stratification S on B such that (B, S) is a K^*-stratified toroidal variety. If $B \setminus (U \times K^*)$ is a normal crossings divisor then the components of this divisor are the closures of some strata in S.

Step 4 By Proposition 12.1.2 we can find a π-regular collapsible K^*-stratified cobordism $(\overline{B}, \overline{R})$ between X and Z by applying a sequence of blow-ups at stable valuations to (B, S) and using the combinatorial algorithm. The subsets $B_+ = \overline{B}_+$ and $B_- = \overline{B}_-$ are unaffected.

Step 5 By Proposition 12.2.1 the π-regular cobordism $(\overline{B}, \overline{R})$ determines a factorization of the map $Z \to X$ into a sequence of blow-ups and blow-downs. All centers have only normal crossings with closures of the induced strata on intermediate varieties. $\qquad\square$

13. COMPARISON WITH ANOTHER PROOF. TORIFICATION AND STABLE SUPPORT

13.1. **Comparison with another proof.** In [4] another proof of the Weak Factorization Theorem is given , where the theorem is stated an proved in a more general version. In particular the assumption of the base field to be algebraically closed was removed. The theorem was also formulated and proved for bimeromorphic maps of complex compact analytic manifolds. The proof in [4] uses the idea of torific ideals due to Abramovich and De Jong. The idea of the proof is to consider an ideal on a smooth cobordism whose blow-up determines a structure of toroidal embedding compatible with the K^*-action. Such an ideal, called *"torific"*, and the the blow-up procedure , called *torification* were introduced by D.Abramovich and J.de Jong. The torific ideal is defined in an invariant neighborhood U of the fixed point component of the smooth variety with K^*-action. Let $u_1, \ldots, u_k$ denote the semiinvariant parameters with weights $a_1, \ldots, a_k$. We define the torific ideal to be $I = I_{a_1} \cdot \ldots \cdot I_{a_k}$, where I_a denotes the ideal generated by all semiinvariant functions of weight a. The

blowing-up of the torific ideal induces a structure of toroidal embedding, locally on each elementary piece of cobordism. Torification allows one to pass to the category of toroidal embeddings, where the factorization problem has already been solved using combinatorial algorithms. However the torific ideal can be constructed only locally on elementary cobordisms. Hence after the torification is done on each elementary cobordism some glueing procedure should be applied. Using canonical principalizations of torific ideals and canonical resolution of singularities of intermediate varieties gives a factorization on an elementary cobordism between canonical resolution of singularities. Patching these factorizations together one gets a decomposition of the given birational map.

13.2. **Torification and stable support.** The notion of torification can be understood and generalized on the ground of the theory of stratified toroidal varieties.

Definition 13.2.1. Let (X, S) be an oriented stratified toroidal variety (with or without group action). By a *stable ideal* $\mathcal{I}$ we mean a coherent sheaf of ideals $\mathcal{I}$ satisfying the following conditions :

1. The support of $\mathcal{I}$ is a union of strata in S.
2. For any points x_1 and x_2 in a stratum s and any isomorphism $\phi : \widehat{X}_{x_1} \to \widehat{X}_{x_2}$ preserving strata and orientation (respectively equivariant with respect to the Γ_σ-action), $\phi^*(\mathcal{I}) \cdot \widehat{\mathcal{O}}_{x_2} = \mathcal{I} \cdot \widehat{\mathcal{O}}_{x_1}$.

Proposition 13.2.2. *Let $\mathcal{I}$ be a stable ideal on an oriented stratified toroidal variety (X, S). Then for any $\sigma \in \Sigma$ there is a G^σ-invariant toric ideal I_σ on X_σ such that*

1. *The induced ideal on $\widetilde{X}_\sigma$ is G^σ-invariant.*
2. *For any chart $\phi_\sigma : U \to X_\sigma$, $\mathcal{I}_U = \phi_\sigma^*(I_\sigma)$.*
3. *For any $\tau \le \sigma$, $I_\tau \cdot \mathcal{O}_{x, X_\sigma}$ is the restriction of I_σ to the open subset $X_{(\tau, N_\sigma)} \subset X_\sigma$.*

One can rephrase the above proposition in combinatorial language, using the correspondence between toric ideals $\mathcal{I}$ and $\mathrm{ord}(\mathcal{I})$-functions in [39] and Lemma 7.6.4.

Proposition 13.2.3. *Let (X, S) be an oriented stratified toroidal variety with an associated oriented semi-complex Σ. The stable complete coherent sheaves of ideals $\mathcal{I}$ on (X, S) are in $1 - 1$ correspondence with collections of functions $\mathrm{ord}(I_\sigma)$, $\sigma \in \Sigma$, satisfying:*

1. $\mathrm{ord}(I_\sigma) : \sigma \to \mathbb{Q}$ *is a convex piecewise linear function $\mathrm{ord}(I_\sigma)(N_\sigma) \to \mathbf{Z}$.*
2. *Denote by $\Delta(I_\sigma)$ the subdivision of σ into the maximal cones where $\mathrm{ord}(I_\sigma)$ is linear. Then $\mathrm{Vert}(\Delta(I_\sigma)) \subset \mathrm{stab}(\sigma) \cup \mathrm{Vert}(\overline{\sigma})$, and $\mathrm{ord}(I_\sigma)(v) = 0$ for any $v \in \mathrm{Vert}(\Delta(I_\sigma)) \setminus \mathrm{stab}(\sigma)$.*
3. *For any $\tau \le \sigma$, the restriction of $\mathrm{ord}(I_\sigma)$ to τ is equal to $\mathrm{ord}(I_\tau)$.* $\qquad\square$

It follows that the functions $\mathrm{ord}(I_\sigma)$ patch together and give a function $\mathrm{ord}(\mathcal{I})$ on the totality of vectors in faces of Σ.

Lemma 13.2.4. *Denote by $\Delta(\mathcal{I})$ the subdivision of Σ obtained by glueing the subdivisions $\Delta(I_\sigma)$. Then $\Delta(\mathcal{I})$ is a canonical subdivision of Σ correspondinding to the blow-up of (X, S) at $\mathcal{I}$.* $\qquad\square$

Definition 13.2.5. Let (X, S) be an oriented stratified toroidal variety (with or without group action). Then an ideal $\mathcal{I}$ is called *torific* if it is stable and the normalization of its blow-up is a canonical toroidal morphism $(Y, R) \to (X, S)$ such that (Y, R) is a toroidal embedding. Such a blow-up is called a *torification*.

Lemma 13.2.6. *Let (X, S) be an oriented stratified toroidal variety with an associated semicomplex Σ. Then a stable ideal $\mathcal{I}$ is torific iff any $\sigma \in \Delta(\mathcal{I})$ whose relative interior intersects the stable support $\mathrm{Stab}(\Sigma)$, is contained in $\mathrm{Stab}(\Sigma)$.*

Proof. If $\mathrm{int}(\sigma) \cap \mathrm{Stab}(\Sigma) \ne \emptyset$ then $\sigma \in \Delta(\mathcal{I})_{\mathrm{stab}}$. By Theorem 6.6.1 and Lemma 4.8.2, $\Delta(\mathcal{I})_{\mathrm{stab}}$ is a complex corresponding to the toroidal embedding. Hence all vectors from σ are stable. $\qquad\square$

Torification is not always possible. However we can perform torification locally in the following sense.

Proposition 13.2.7. *Let (X, S) be an oriented stratified toroidal variety with an associated oriented semi-complex Σ. Then for any $\sigma \in \Sigma$ there is a G^σ-invariant ideal*

$$I_\sigma := (G^\sigma \cdot u_1) \cdot \ldots \cdot (G^\sigma \cdot u_k),$$

where $u_1, \ldots, u_k$ are toric paramaters on $\widetilde{X}_\sigma$ such that $\Delta(I_\sigma)$ contains the face $\mathrm{Inv}(\sigma)$. In particular

$$\mathrm{Inv}(\sigma) = \mathrm{conv}((\mathrm{Vert}(\Delta(I_\sigma)) \setminus \mathrm{Vert}(\sigma)) \cup \mathrm{Vert}^{\mathrm{semic}}(\sigma)),$$

where $\mathrm{Vert}^{\mathrm{semic}}(\sigma)$ *denotes the set of 1-dimensional faces in the semicone* σ.

Proof. Denote by v_ν the vector corresponding to a toric valuation ν. Note that a toric valuation ν is G^σ-invariant iff $\nu(g_*(u_i)) = \nu(u_i)$ for any $i = 1, \ldots, k$ and $g \in G^\sigma$. It follows from Lemma 5.1.4 that the latter statement is equivalent to the following one: $\nu(g_*(u_i)) \geq \nu(u_i)$ for any $g \in G^\sigma$. This can be reformulated as follows: ν is G^σ-invariant iff $\mathrm{ord}(I_\sigma)(v_\nu) = \nu(u_1 \cdot \ldots \cdot u_k)$. Finally, $\mathrm{Inv}(\sigma)$ is a set where $\mathrm{ord}(I_\sigma)$ is linear and equal to the functional of $u_1 \cdot \ldots \cdot u_k$. Hence $\mathrm{Inv}(\sigma)$ occurs as a cone in $\Delta(I_\sigma)$. All vertices of this cone belong to $\mathrm{Inv}(\sigma)$ and hence they have to be in the form as in the proposition. $\square$

Remark. The above proposition generalizes the construction of torific ideal due to Abramovich and de Jong ([2]).

13.3. **Computing the stable support.** Proposition 13.2.7 provides a method of computing the stable support of semicomplexes.

Example 13.3.1. Let (X, S) be a toroidal variety with an isolated singularity $x_1 x_2 = x_3 x_4$ as in Example 4.10.7. The associated semicomplex Σ consists of the cone over a square and the apex of this cone. Then $G^\sigma \cdot x_i = (x_1, x_2, x_3, x_4)$ is the ideal m_p of the singular point p. Hence the torific ideal I_σ, where $s = \{p\}$, is equal to m_p^4. $\Delta(I_\sigma)$ is the star subdivision at the ray ϱ over the centre of the square. By Proposition 13.2.7,

$$\mathrm{Stab}(\Sigma) = \mathrm{Inv}(\sigma) = \varrho.$$

Example 13.3.2. Let (X, S) be a smooth stratified toroidal variety of dimension 3 with the stratification determined by two curves l_1 and l_2 meeting transversally at some point p. The associated semicomplex Σ consists of the regular 3-dimensional cone $\sigma = \langle w_1, w_2, w_3 \rangle$ and its two 2-dimensional faces $\tau_1 = \langle w_1, w_3 \rangle$ and $\tau_2 = \langle w_2, w_3 \rangle$. Set $v_1 := w_1 + w_3$, $v_2 := w_2 + w_3$, $v := w_1 + w_2 + w_3$. Write $l_1 : x_1 = x_3 = 0$, $l_2 : x_2 = x_3 = 0$. Then $(G^\sigma \cdot x_1) = I_{l_1} = (x_1, x_3)$, $(G^\sigma \cdot x_2) = (x_2, x_3)$, $(G^\sigma \cdot x_3) = I_{l_1 \cup l_2} = (x_3, x_1 x_2)$. Then $\Delta(I_\sigma)$ is determined by two star subdivisions: $\langle v_1 \rangle \cdot \sigma$, $\langle v_2 \rangle \cdot \sigma$, and the subdivision Σ_0 of σ determined by $\mathrm{ord}((x_3, x_1 x_2)) := \min(w_3^*, w_1^* + w_2^*)$. The latter consists of two cones separated by the 2-dimensional face $\langle v_1, v_2 \rangle$. Then $\mathrm{Inv}(\sigma) = \langle v_1, v_2, v \rangle$. Also $\mathrm{Inv}(\tau_i) = \langle v_i \rangle$. Finally,

$$\mathrm{Stab}(\Sigma) = \mathrm{Inv}(\sigma) = \langle v_1, v_2, v \rangle.$$

The above considerations can be generalized to any regular semicomplex. We shall skip the details of the proof.

Proposition 13.3.3. *Let* Σ *be a regular semicomplex. For any cone* $\sigma = \langle v_1, \ldots, v_{k_\sigma} \rangle$ *set* $v_\sigma := v_1 + \ldots + v_{k_\sigma}$. *Then*

$$\mathrm{stab}(\sigma) = \bigoplus_{\tau \leq \sigma} \mathbf{Q} \cdot v_\tau.$$

$\square$

This formula can be extended to any oriented simplicial semicomplex.

Conjecture 13.3.4. *Let* Σ *be an oriented simplicial semicomplex. For any* $\sigma \in \Sigma$ *let* V_σ *denote the set of all the minimal internal vectors of* σ *and* $W_\sigma := \bigcup_{\tau \leq \sigma} V_\tau$. *Then*

$$\mathrm{stab}(\sigma) = \bigoplus_{v \in W_\sigma} \mathbf{Q} \cdot v.$$

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